Consistently Negative Feedback

Peter Moylan

Glendale, NSW

peter@pmoylan.org http://www.pmoylan.org

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ABSTRACT: We introduce the concept of "consistently negative feedback" (CNF) in a control system. This is a generalisation of the property of a linear system where the feedback is negative at all frequencies. A number of properties of CNF systems are derived.

1. Introduction

Historically, control and systems theory began with the observation that *feedback* is an important property. It was observed, at a fairly early stage, that negative feedback made a system more stable, and positive feedback made it less stable. In a linear system, it can be the case that the feedback is positive at some frequencies and negative in others. But what if the feedback is negative at every frequency? Does this give some extra desirable properties? That is the subject of this paper, but we will express it in terms that don't require linearity.

For convenience, time-invariant systems will be assumed. That is not an essential assumption, but it simplifies the notation. Otherwise, we can allow linear or nonlinear systems, not necessarily finite-dimensional, evolving over either discrete or continuous time. We do assume a finite number of input and output ports, and we also need to assume that the system is causal.

This paper relies heavily on results for (Q, S, R) dissipative systems. Those results can be found in [6], which in turn uses results from [2, 3, 4].

2. The CNF property

A standard feedback loop is shown in Figure 1. The components are the plant to be controlled, a controller, and optionally an estimator. The estimator is needed only if the outputs of the plant must be processed in some way to obtain information needed by the controller.

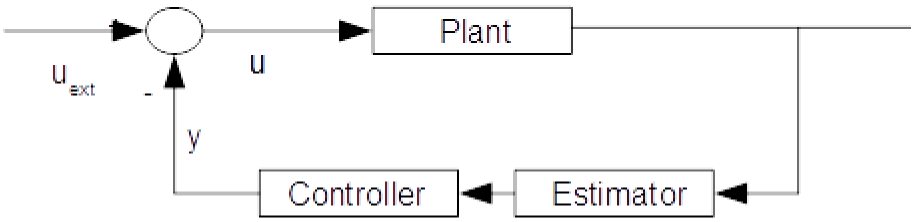

Fig 1 Single-loop feedback

For our present purposes we don't need that level of detail, so we can simplify this down to a unity-feedback loop, as shown in Figure 2. The operator G represents the concatenation of everything in the loop: the plant plus all control elements. By time-invariance, we can assume a fixed initial time of 0.

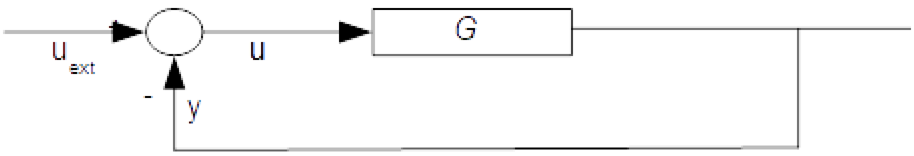

Fig 2 The basic loop

In what follows, a symmetric positive definite matrix R will be used to specify a weighting on the system inputs. For a single-input system, there is no loss of generality in setting $R = 1$. Otherwise, R can be used to specify the relative importance of the inputs. If there is no particular reason to scale the inputs, then we can set $R = I$.

Now we come to the crucial definition. We call this a **Consistently Negative Feedback** (CNF) loop, relative to the weighting R, if

$$\int_0^T u(t)'Ru(t)dt \leq \int_0^T u_{ext}(t)'Ru_{ext}(t)dt$$

for all possible inputs, and all $T \geq 0$, where the superscript prime denotes matrix or vector transpose. That is for the continuous-time case. For a discrete-time system, the inequality becomes

$$\sum_{t=0}^{T} u(t)'Ru(t) \leq \sum_{t=0}^{T} u_{ext}(t)'Ru_{ext}$$

We can express this more conveniently in the language of inner products and norms. Following [6], we can define a causal truncation operator P_T, for any real time $T \geq 0$, by

$$(P_T f)(t) = \begin{cases} f(t) & \text{for } t < T \\ 0 & \text{for } t \geq T \end{cases}$$

(Note that this works in both continuous and discrete time.) Define an inner product by

$$< u, v >= \int_0^\infty u(t)'Rv(t)dt$$

and the induced norm by

$$\|u\| = \sqrt{< u, u >}$$

which are easily shown to be valid inner products and norms if the matrix R is symmetric and positive definite. Finally, define a truncated norm and truncated inner product as

$$\|u\|_T = \|P_T u\|$$

$$< u, v >_T = < P_T u, P_T v > = < u, P_T v > = < P_T u, v >$$

where the last sequence of equalities follows from the obvious fact that P_T is an idempotent and self-adjoint operator.

With this background, the CNF definition becomes

$$\|u\|_T \leq \|u_{ext}\|_T$$

for all u and all $T \geq 0$. This works equally well in continuous time and discrete time.

Consider the meaning of this last inequality. It says that the input u to the controlled plant is never bigger, in an "energy" sense, than the external input. That is, the feedback reduces the input – or, at worst, leaves it equal in size – under all conditions. That is what we mean by negative feedback.

For a purely linear system, it is not hard to show that this is equivalent to the feedback being negative (or at least nonpositive) at all frequencies.

In the language of dissipative systems [6], the condition is that the map from u_{ext} to u is $(-I, 0, I)$ dissipative. That is, it is a finite-gain map, with a gain bound of 1.

In practice, we are more interested in the map from u to y (the open-loop system) and the map from u_{ext} to y (the closed-loop system). From the equation

$$u = u_{ext} - y$$

we easily see that

$$< y, y >_T + 2 < u, y >_T \geq 0$$

and

$$-< y, y >_T + 2 < u_{ext}, y >_T \geq 0$$

This tells us that

(a) the open-loop system is $(I, I, 0)$ dissipative, and
(b) the closed-loop system is $(-I, I, 0)$ dissipative.

Let us now ask what further properties those observations imply.

3. Stability

The main result of [2] is that a (Q, S, R) dissipative system is stable if $Q < 0$. It follows that a CNF system is always closed-loop stable. Of course we must qualify this by requiring well-posedness properties that rule out uncontrollable or unobservable unstable modes that are invisible in an input-output description, but that is a standard qualification.

We cannot, of course, guarantee that the open-loop system is stable. That was hardly to be expected.

Since CNF is a strong form of negative feedback, the stability result is hardly surprising. Still, it is an important preliminary to some less obvious properties.

When we refer to stability in this paper, we mean input-output stability, defined in terms of a square-integral norm. It is known [5] that, given suitable assumptions, this also implies asymptotic stability in the sense of Lyapunov. There are borderline cases where a system can be stable, but not asymptotically stable, in the sense of Lyapunov, but we shall not attempt to cover those cases here.

4. Gain margin

The gain margin of a loop tells how much we can modify the loop gain, and still retain stability. Let us therefore consider the loop of Figure 3, where K is a constant gain.

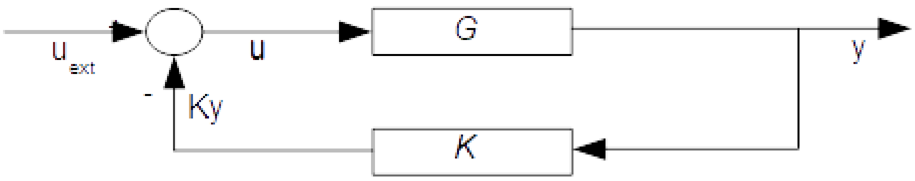

Fig 3 Modified feedback gain

The forward path G is still a subsystem that is $(I, I, 0)$ dissipative, so we have

$$< y, y >_T + 2 < y, u >_T \geq 0$$

But now $u = u_{ext} - Ky$, therefore

$$-(2K - 1) < y, y >_T + 2 < y, u_{ext} >_T \geq 0$$

The closed-loop system is $(-(2K - 1), I, 0)$ dissipative, so it is stable if the constant K is in the range $\left(1/_2, \infty\right)$. That is, it remains stable if the loop gain drops by a factor of up to 2, and also if the loop gain is increased indefinitely. That is, a CNF loop has a very generous gain margin.

5. Relationship to passivity

Consider the special case, in the previous section, where $K = \frac{1}{2}$. In this case, the closed-loop system is $(0, I, 0)$ dissipative. That is, it is passive. A CNF system therefore has the property that it is passive, and remains passive even when the loop gain is halved.

Interestingly, the converse is also true. If the closed-loop system is passive when $K = \frac{1}{2}$, then

$$< u + \frac{1}{2} y, y >_T \geq 0$$

and therefore

$$< y, y >_T + 2 < u, y >_T \geq 0$$

That is, the open-loop system is $(I, I, 0)$ dissipative. We have already seen that this is equivalent to the defining property for a CNF loop. We conclude, therefore, that *a unity feedback loop is CNF iff the same system with its loop gain halved is passive*. Or, equivalently, a CNF loop is one that has twice the loop gain that would be required to make it passive.

A difference between this result and that of Section 4 is that, for the stability result, we want the gain to be strictly greater than one half. In the borderline case $K = \frac{1}{2}$ we can guarantee passivity, but the resulting system might or might not be stable. For a linear system, the borderline case is where we cannot rule out a pole on the imaginary axis.

6. Tolerance of feedback nonlinearities

Let us now consider a system of the form of Figure 3, but where K is no longer required to be a constant. This is clarified in Figure 4, where the label NL is used to clarify that the extra element in the feedback path is possibly nonlinear. (And not necessarily memoryless.)

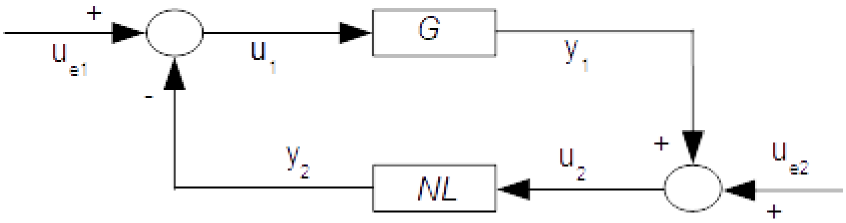

Fig 4 A sector nonlinearity in the feedback path

The interconnection equation is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_{e1} \\ u_{e2} \end{bmatrix} - \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The additional external input is a technicality required to use the stability criteria of [3] and [4], but of course it can be set to zero after doing the stability analysis.

If subsystem NL is a sector nonlinearity in the sector $\left(\frac{1}{2}, \infty\right)$, it obeys the constraint

$$< u_2, y_2 - \frac{1}{2} u_2 > \geq 0$$

but this does not rule out the boundary case where the gain is $1/2$. To turn the half-closed interval into an open interval, we can modify this to

$$< u_2, y_2 - \frac{1}{2} u_2 > \geq \delta < u_2, u_2 > + \epsilon < y_2, y_2 >$$

for some small $\delta > 0$ and $\epsilon > 0$.

Applying the stability test of [3, Theorem 2] or [4, Theorem 1], we deduce that the closed-loop system is stable if

$$\hat{Q} = \begin{bmatrix} Q_1 + \alpha R_2 & -S_1 + \alpha S_2 \\ -S_1' + \alpha S_2' & R_1 + \alpha Q_2 \end{bmatrix}$$

is negative definite for some $\alpha > 0$. In our case, the parameters are $(Q_1, S_1, R_1) = (I, I, 0)$ and $(Q_2, S_2, R_2) = (-\epsilon I, \frac{1}{2}I, -\frac{1}{2}I - \delta I)$, which gives

$$\hat{Q} = \begin{bmatrix} I - \frac{\alpha}{2}I - \alpha\delta I & -I + \frac{\alpha}{2}I \\ -I + \frac{\alpha}{2}I & -\alpha\epsilon I \end{bmatrix}$$

This is negative definite if we set $\alpha = 2$.

7. Tolerance of nonlinearities in the input sensors

Figure 5 shows a variation of this theme, with the sector nonlinearity moved to a different part of the loop. The difference is that the interconnection equations are now

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_{e1} \\ u_{e2} \end{bmatrix} - \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

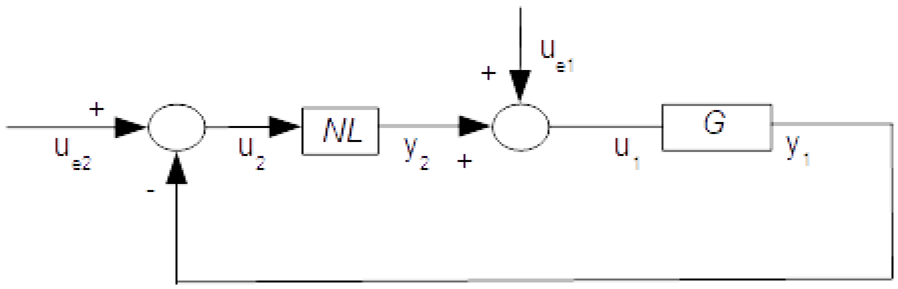

Fig 5 A sector nonlinearity at the plant input

Let H be the interconnection matrix in that equation. Applying the reasoning of [4], but with a sign change to be consistent with the Section 6 calculations, we now have to calculate

$$\hat{Q} = Q - SH - H'S' + H'RH$$

where Q, S, and R are as defined in that reference. The scaling found in the last section suggests that we should double the subsystem NL parameters. This gives us

$$\hat{Q} = \begin{bmatrix} I & 0 \\ 0 & -2\epsilon I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -I - 2\delta I \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

which reduces to

$$\hat{Q} = \begin{bmatrix} -2\delta I & 0 \\ 0 & -2\epsilon I \end{bmatrix}$$

It should not be too surprising that this is the same final result as in the last section, because the loop "gain" is the same in both cases. We conclude, then, that a CNF system remains stable with the sector nonlinearity in either position.

8. A connection with optimal control

Our next result is rather less obvious, but in fact it is a simple reinterpretation of a result that was established some time ago. Reference [1] deals with systems with state equations of the form

$$\frac{dx}{dt} = f(x(t)) + Gu(t)$$

and an optimal control problem where we must minimise a performance index

$$V = \int_0^\infty \Big(m\big(x(t)\big) + u(t)'Ru(t)\Big)\,dt$$

where R is a positive definite symmetric matrix, and $m(.)$ is a function with the property that $m(0) = 0$ and $m(x) \geq 0$ for all x. (To be technically correct, the infinite-time problem must be posed as the limiting case of a finite-time optimisation, but for our present purposes we can afford to gloss over such detail.) The fact that the performance index is quadratic in the control, but not necessarily in the state, moved the problem one step beyond known results for the well-known linear-quadratic optimisation problem.

The optimal control for this problem can be expressed as a feedback law

$$u^*(t) = -k(x(t))$$

where the form of $k()$ does not concern us here. Of special interest here is the *inverse problem*: given a feedback law of this form, when is this feedback law optimal, in the sense that there exists an $m()$, with $m(0) = 0$ and $m(x) \geq 0$ for all x, such that this feedback law is a solution to the originally posed optimisation problem? The conclusion in [1] is that such a law is optimal iff $k()$ satisfies something called the Return Difference Condition (RDC)

$$\int_0^\infty \big(u + k(x)\big)'\big(u + k(x)\big)dt \geq \int_0^\infty u'u\,dt$$

whenever $x(0) = 0$, for any u such that $x(\infty) = 0$. The term "return difference condition" comes from an earlier known result for linear systems with a quadratic performance index. In that case, the RDC can be expressed in frequency domain terms. This result is for the case $R = I$, but it is obvious how it can be extended to the case of an arbitrary positive definite R.

Let us note that the term $(u + k(x))$ is precisely equal to the external input. That says that we are looking at a CNF condition. Of course we need to reconcile the relationship between an infinite-time integral and a finite-time one, but that point is covered in [1]. The result there does depend on a condition that $k(x)$ is a stabilising feedback law, but for our present purposes that does not matter, because we already know (Section 3) that a CNF system is automatically stable. Thus, we can tie our present results with those of [1] by saying that a feedback law $u = -k(x(t)$ is optimal, for the class of optimisation problems under consideration, iff the resulting closed-loop system is a CNF loop. In other words, the CNF condition is equivalent to a certain kind of optimality.

Can we extend this to a larger class of optimisation problems? Probably not. The fact that the performance index is quadratic in the control, and the state equations are linear in the control, appears to be fairly central to the results cited here.

9. Sensitivity

A known result from linear-quadratic control theory is that optimal control leads to decreased parameter sensitivity. Can we claim the same for CNF systems? Probably not. In nonlinear systems, dealing with sensitivity issues seems to require looking at the locally linearised form of the nonlinear system. That goes beyond the scope of the issues explored in this paper.

10. Conclusions

It is generally acknowledged that negative feedback is a Good Thing. The goal of this paper is to pin down that idea more precisely. A CNF system, one where the feedback is always negative, turns out to have a number of desirable properties.

The connection with optimal control is a bonus. It shows that optimal control is also a Good Thing, because it leads to the same set of desirable properties.

References

[1] P.J. Moylan and B.D.O. Anderson. Nonlinear regulator theory and an inverse optimal control problem. IEEE Trans Automat Contr, AC-18(5):460{464, Oct 1973.

[2] D.J. Hill, P.J. Moylan, "The stability of nonlinear dissipative systems", IEEE Trans Automat Control AC-21 (5), Oct 1976, 708-711. Also Tech Rept EE7518, University of Newcastle, Aug 1975.

[3] D.J. Hill and P.J. Moylan. Stability results for nonlinear feedback systems. Automatica, 13:377–382, 1977.

[4] P.J. Moylan and D.J. Hill. Stability criteria for large-scale systems. IEEE Trans Automat Contr, AC-23(2):143–149, Apr 1978.

[5] D.J. Hill and P.J. Moylan. Connections between finite gain and asymptotic stability. IEEE Trans Automat Contr, AC-25(5):931{936, Oct 1980

[6] P.J. Moylan, *Dissipative Systems and Stability*, http://www.pmoylan.org/pages/research/DissBook.html, August 2014.