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STABILITY OF LOCALLY-DISSIPATIVE INTERCONNECTED SYSTEM

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Abstract

This short note is concerned with a method for checking stability of large-scale interconnected systems. Here we take the dissipativity approach because our interest is in establishing results for both internal and external behaviour of the interconnected system. The novelty of the result reported in here lies in the ability to handle subsystems whose properties are known "locally" and not globally. The result has an immediate application in studying the power system transient stability problem and can have possible applications in robust system analysis.

1. Introduction

The two, internal or state-space and external or input-output, representations are widely used for stability analysis and controller design. Depending on the situation either one can be useful so attempts [1] have been made to transfer analysis done on one representation to another. The analysis in [1] is for the case where all the properties hold globally. In [2] an attempt is made to do the same for systems where the required properties hold only locally. Unfortunately the analysis in [2] is not very helpful when we intend to analyse an interconnection of these "locally" known system. A large power system is a good example of an interconnection of locally stable systems.

In this short note an attempt is made to develop a frame work for the analysis of locally-dissipative interconnected systems. The interconnection of dissipative systems is covered in [3] along with a relationship between internal and external stability [4]. Section 2 gives the basic notations and definitions. Section 3 gives the main result of this paper and section 4 states the conclusions.

2. Notation and Definitions

Let U be an inner product space whose elements are functions u: $\mathcal{R} \to \mathcal{R}$. Also let U^n be the space of n-tuples (column vectors) over U, with the usual inner product, and the "truncated inner product" generating the extended space U_e^n as defined in [3].

A system with m inputs and p outputs may now be formally defined as a relation on $U_e^m \times U_e^p$; that is, a set of pairs $\left(u \in U_m^e,\ y \in U_p^e\right)$, where u is an input and y the corresponding output. We also assume that there exists a state space X for the dynamic system [3].

<u>Definition</u> A dynamic system is *locally (Q, S, R) dissipative in a region* $\Omega \subseteq X$, if

$$\int_0^T w(u,y)\,dt = \langle y,Qy\rangle_T + 2\langle y,Su\rangle_T + \langle u,Ru\rangle_T \ge 0$$

for all $u \in U_e$ and $T \in \mathcal{R}^+$, such that x(0) = 0 and $x(t) \in \Omega$ for all $0 \le t \le T$. Q, S and R are constant matrices of appropriate dimensions.

To proceed further we need the notation:

$$U(t_o, t_1, x_o, x_1) \overset{\Delta}{=} \left\{u \in U_e : x(t_o) = x_o, x(t_1) = x_1, x(t) \in \Omega,\ t_o \le t \le t_1\right\}$$

$U(t_o, t_1, x_o, x_1)$ denotes all the inputs belonging to the extended space U_e, taking the state from x_0 to x_1 and keeping it in a specified region Ω for the entire duration $[t_0, t_1]$.

<u>Definition</u> A state $x_1 \in \Omega$ is said to be *locally reachable* wrt Ω from $x_0 \in \Omega$ at time t_1 if $\exists\ t_0 \le t_1$ such that $U(t_0, t_1, x_0, x_1)$ is non empty. A region $\Omega_R \subseteq \Omega$ of the state space of a dynamic system is said to be locally reachable with respect to Ω if every state $x_1 \in \Omega_R$ is locally reachable wrt Ω from the origin for all $t_0 \in \mathcal{R}$. A region $\Omega_c \subseteq \Omega$ of a dynamic system is said to be *locally connected* wrt Ω if any $x_1 \in \Omega_c$, is locally reachable wrt Ω from any other $x_0 \in \Omega_c$, for all $t_0 \in \mathcal{R}$.

<u>Definition</u>. A function $\phi: \Omega \times \mathcal{R} \to \mathcal{R}$ is called a storage function if it satisfies $\phi(o, t) = 0$ for all t and

$$\phi(x_o, t_o) + \int_{t_o}^{t_1} w(u, y)\,dt \ge \phi(x_1, t_1)$$

for all $t_1 \ge t_0$ and all $u \in U(t_0, t_1, x_0, x_1)$ and $\phi(x, t) \ge 0$ for all $x \in \Omega$ and all $t \in \mathcal{R}$.

<u>Definition</u> A dynamic system is said to be *locally uniformly uncontrollable* at $x_0 \in \text{int}(\Omega)$ if, there exists an open neighbourhood B_ϵ of x_0, such that for any $x_1 \in B_\epsilon$ there exist choices of $U \in U_e$ and t_1 such that the state can be driven from $x(t_0) = x_0$ to $x(t_1) = x_1$, $\{x(t) \in \Omega, t_0 \ge t \ge t_1\}$ with the additional property that

$$\left|\int_{t_o}^{t_1} w(u,y)\,dt\right| \le \rho\,(\|x_1 - x_0\|)$$

for some continuous function $\rho: \mathcal{R}^+ \to \mathcal{R}^+$ such that $\rho(0) = 0$. The dynamic system is said to be locally uniformly controllable at every state $x_0 \in \text{int}(\Omega)$.

<u>Definition</u> A dynamic system is said to be *locally zero state detectable* in a region Ω_Z with respect to Ω if, for any $x_0 \in \Omega_Z$, $x_0 \ne o$ such that $x(t) \in \Omega$ for $o \le t \le \delta$ for some $\delta > 0$ with $u(.) \equiv 0$, there exists a continuous function $\alpha(0) = 0$ and $\alpha(\sigma) > 0$ for all $\sigma > 0$ such that

$\int_0^t y^t\, y\, dt \geq \alpha(|x_o|)$, for some finite T such that $0 \leq T \leq \delta$

If in addition, for any sequence $\{\sigma_n\} \in \Omega$, $\alpha(\sigma_n) \to \infty$ as $\|\sigma_n\| \to \infty$ the system is called locally uniformly zero state detectable in Ω_Z with respect to Ω.

Definition A dynamic system is said to have a *property A* if there exists a well defined feedback law $u^*(\cdot)$ such that $w(u^*(\cdot), y) < 0$ for all $y \neq 0$, $u^*(0) = 0$ and $u^* \in U(t_0, t_1, x_0, x_1)$.

3. Main results

First we will put conditions on the system and its local properties such that it is internally stable.

Theorem 1 Let the dynamic system

(i) be locally (Q, S, R) - dissipative in a region $\Omega \subseteq X$; (ii) be locally connected in a region Ω_c wrt Ω; (iii) locally uniformly controllable in a region Ω_Z wrt Ω; (iv) locally uniformly zero state detectable in region Ω_Z wrt Ω; (v) locally Lipschitz continuous in the region Ω; (vi) have the property A. Furthermore, suppose that the region $\Omega_S \triangleq \Omega_c \cap \Omega_{uc} \cap \Omega_Z$ is non-empty, in the sense that it contains an open neighbourhood of the origin. Then if *Q is negative definite* then the origin is asymptotically stable.

Proof: See [5. pp. 48-50]

Remark 1: The proof proceeds more or less on the standard lines as in [3]. The difference is only in the fact that we need here only *local* description as opposed to the global needed elsewhere. During the course of a proof we need a *storage function* ϕ (x) which serves as a Lyapunov function, needed to prove the asymptotic stability.

The next theorem refers to a linear interconnection of N locally-dissipative subsystems. The interconnection is described by.

$$U_i = U_{ei} - \sum_{j=1}^{N} H_{ij} y_j,\ i = 1,\ldots,n \qquad (1)$$

Where U_i is the input to subsystem i, y_i is the output, U_{ei} is an external input, and the H_{ij} are constant matrices. A compact matrix notation is, with obvious definitions, $U = U_e - Hy$

Theorem 2 Let the dynamic system be formed by interconnecting N subsystems via the interconnection (1) and suppose that (i) The i^{th} subsystem is locally (Q_i, S_i, R_i) - dissipative in a region Ω_i satisfying conditions *(i)-(vi) of Theorem 1.* (ii) The interconnection (1) is such that the dynamic system state space $\hat{X}$ equal to the cartesian product of the state space of the individual subsystems and $\Omega = \Omega_1 \times \ldots \times \Omega_N$ is a non-empty region containing a neighbourhood of the origin ($\Omega \subseteq \hat{X}$). (iii) The overall system is uniformly zero state detectable in a region Ω_Z with respect to Ω, where Ω_Z is a non-empty region containing a neighbourhood of the origin ($\Omega_Z \subseteq \hat{X}$). (iv) The overall system is locally Lipchitz continuous in Ω.

Then if $\hat{Q}$ *is positive definite*, the origin is asymptotically stable

Where $\hat{Q} = SH + H^T S^T - H^T RH\text{-}R$ and Q = diag. (Q_i), S = diag. (s_i), R = diag. (R_i).

Proof: See [5. pp. 50-52]

Remark 2: The proof is similar to the one in [3]. Again the concepts are *local* instead of global. The above two theorems when put together give a method of constructing Lyapunov functions for an interconnected system. The steps to follow are;

1. Decompose a large-scale system into N subsystems.
2. Find out the *storage function* $\phi_i(x_i)$, and the corresponding (Q_i, S_i, R_i) for each subsystem, using methods discussed in [5].
3. Form the matrix $\hat{Q} = SH + H^T S^T - H^T RH\text{-}Q$ and check for its positive definiteness. If $\hat{Q}$ is positive definition then $\sum_{i=1}^{N} \phi_i(x_i)$ is a Lyapunov function. If $\hat{Q}$ is not positive definite, then it becomes necessary to try another decomposition or different choice of (Q_i, S_i, R_i). The above three steps are successfully used to construct Lyapunov functions for large power systems [5].

4. Conclusion

The main result stated in Theorem 2 is just the type of result we need to study the behaviour of an interconnected system whose subsystems are only "locally" defined. To obtain the result we have made precise what is "locally" required of the sub systems. If the definitions are read carefully one can see that "local" is both in terms of small gain inputs [2] and local internal stability regions. Hence, this short paper can be considered as a contribution in extending the already available results of [1], [2] [4]. This extension has an immediate application for transient stability analysis of power systems [5].

References

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