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Stability of Locally Dissipative Interconnected Systems

H. R. Pota and P. J. Moylan

Abstract—**This note extends the results of the existing dissipative systems theory to include locally dissipative systems. This extension enables us to analyze an interconnection of locally dissipative systems. The dissipative systems theory has been successful in unifying the input–output analysis and the state-space analysis; so far, a "local" version of this analysis is missing. In this note we firstly give a definition of a locally dissipative system, and then develop results to analyze interconnections of these locally dissipative systems. The result has an immediate application in studying the power-system transient stability problem and can have possible applications in robust system analysis.**

I. Introduction

The two representations—internal or state-space and external or input–output—of dynamical systems are widely used for stability analysis and controller design. Depending on the situation, either one can be useful, so attempts [1] have been made to transfer analysis done on one representation to another. The analysis in [1] is for the case where all the properties hold globally. In [5], an attempt is made to do the same for systems where the required properties hold only locally. Unfortunately the analysis in [5] is not very helpful when we intend to analyze an interconnection of these locally stable systems (individual generators) where we wish to know about the stability properties of the overall system.

The dissipativity systems theory [2], [7]–[9], [13], [6], provides a nice framework to unify the results obtained in these two—input–output and state-space—seemingly different settings. In [14], results are reported that use the passivity framework (a special case of the dissipative framework) to unify various results in the area of stabilization of nonlinear systems. In this note, an attempt is made to extend the dissipative systems theory to include locally dissipative systems; with a view to analyzing the locally dissipative interconnected systems. Locally dissipative systems can be thought of as input–output representation analog of the local state-space theory. An important contribution of this note is in obtaining stability results for the interconnection of locally dissipative systems. This extension has been facilitated by a careful definition of the "local" concepts, which can be seen as a generalization of the small-signal input–output [5] concepts. This is similar to the generalization of the finite-gain input–output concept by the existing dissipative systems theory.

The development of the theory in this note is similar to the presentation of the interconnection of dissipative systems in [6] and the relationship between internal and external stability in [8]. Section II gives the basic notations and definitions. Section III discusses the locally dissipative systems, and establishes some necessary results on which can be built the structure to study an interconnection of these local dissipative systems. Section IV gives the main result of this note; first it connects the local dissipativity with the local internal stability, and secondly, it states as to when an interconnection of these locally dissipative systems is internally stable along with a procedure to get an estimate of the region of stability. Section V, states the conclusions that can be drawn from the work done so far.

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II. Notation and Definitions

Let U be an *inner product* space whose elements are functions $(\cdot)$: $\boldsymbol{R} \rightarrow \boldsymbol{R}$. Also let U^n be the space of n-tuples (column vectors) over U, with inner product

$$\langle u, v \rangle = \sum_{i=1}^{n} \langle u_i, v_i \rangle.$$

Then for any $u \in U^n$ and any $T \in \boldsymbol{R}$, a truncation u_T can be defined via

$$u_T(t) = \begin{cases} u(t) & \text{for } t < T \\ 0 & \text{otherwise.} \end{cases}$$

It is also useful to speak of a "truncated inner product"

$$\langle u, v \rangle_T = \langle u_T, v_T \rangle.$$

Finally, let us define an extended space $U_e^n = \{u | u_T \in U^n$ for all $T \in \boldsymbol{R}\}$.

A system with m inputs and p outputs may now be formally defined as a relation on $U_e^m \times U_e^p$; that is, a set of pairs $(u \in U_e^m, y \in U_e^p)$, where u is an input and y the corresponding output. For the present, we assume that there exists a state space X for the dynamic system, and we need no more information than that. A formal definition of the state space X is given in [6].

Definition 1: A dynamical system is locally (Q, S, R)-dissipative in a region $\Omega \subseteq X$, if

$$\int_0^T w(u, y)\, dt = \langle y, Qy \rangle_T + 2\langle y, Su \rangle_T + \langle u, Ru \rangle_T \geq 0.$$

$\forall u \in U_e$ and $T \in \boldsymbol{R}^+$, such that $x(0) = 0$ and $x(t) \in \Omega$ for $0 \leq t \leq T$. $Q \in \boldsymbol{R}^{p \times p}$, $S \in \boldsymbol{R}^{p \times m}$, and $R \in \boldsymbol{R}^{m \times m}$ are constant matrices; $w(u, y)$ is called the supply rate and is given by

$$w(u, y) = y^T Q y + 2 y^T S u + u^T R u$$

we sometimes also say that the system is dissipative with respect to (wrt) supply rate $w(u, y)$.

There are some standard methods to get the (Q, S, R) matrices such that the system is (Q, S, R)-dissipative. The derivation of the dissipativity parameters for linear systems is discussed in [2], and a general discussion for the nonlinear case is given in [7]. Examples of systems for which the dissipativity parameters are derived using the above-cited references are given in [11] and [12].

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To proceed further, we need the notation:

$$U(t_0, t_1, x_0, x_1) \triangleq \{u \in U_e: x(t_0) = x_0, x(t_1) = x_1, x(t) \in \Omega, t_0 \le t \le t_1\}.$$

$U(t_0, t_1, x_0, x_1)$ denotes all the inputs belonging to the extended space U_e, taking the state from x_0 to x_1, and keeping it in a specified region Ω for the entire duration $[t_0, t_1]$. Concatenation of two inputs $u_0 \in U(t_0, t_1, x_0, x_1)$ and $u_1 \in U(t_1, t_2, x_1, x_2)$ is defined as

$$u_0 \circ u_1 = \begin{cases} u_0(t) & \text{for } t_0 \le t \le t_1 \\ u_1(t) & t \ge t_1. \end{cases}$$

The concatenation $u_0 \circ u_1$ might not be continuous, but the state transition due to this control is.

Definition 2: A state $x_1 \in \Omega$ is said to be locally reachable wrt Ω from $x_0 \in \Omega$ at time t_1 if $\exists t_0 \le t_1$, such that $U(t_0, t_1, x_0, x_1)$ is nonempty. A dynamical system is said to be locally reachable wrt Ω, in a region $\Omega_R \subseteq \Omega$, if every state $x_1 \in \Omega_R$ is locally reachable with respect to Ω from the origin $\forall t_0 \in \boldsymbol{R}$. A dynamical system is said to be locally connected wrt Ω, in a region $\Omega_C \subseteq \Omega$, if any $x_1 \in \Omega_C$ is locally reachable wrt Ω from any other $x_0 \in \Omega_C$, $\forall t_0 \in \boldsymbol{R}$.

Definition 3: A function ϕ: $\Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is called a storage function, for any given $Q \in \boldsymbol{R}^{p \times p}$, $S \in \boldsymbol{R}^{p \times m}$, and $R \in \boldsymbol{R}^{m \times m}$, if it satisfies $\phi(0, t) = 0$, $\forall t$ and

$$\phi(x_0, t_0) + \int_{t_0}^{t_1} w(u, y)\, dt \ge \phi(x_1, t_1) \tag{1}$$

$\forall t_1 \ge t_0$, $u \in U(t_0, t_1, x_0, x_1)$, $\phi(x, t) \ge 0$, $\forall x \in \Omega$, and all $t \in \boldsymbol{R}$.

Further to the general definition of the storage function we introduce two special storage functions to be used later:

i) Required supply as

$$\phi_r(x_1, t_1) = \inf_{\substack{u \in U(t_0, t_1, 0, x_1) \\ t_1 \ge t_0,\, t_0 \in [-\infty, t_1]}} \int_{t_0}^{t_1} w(u, y)\, dt. \tag{2}$$

ii) Available supply as

$$\phi_a(x_0, t_0) = -\inf_{\substack{u \in U(t_0, t_1, x_0, x_1) \\ t_1 \ge t_0,\, t_1 \in [t_0, \infty],\, x_1 \in \Omega}} \int_{t_0}^{t_1} w(u, y)\, dt. \tag{3}$$

III. Locally Dissipative Systems

Lemma 1: Let the dynamical system be locally connected in a region Ω_c wrt Ω:

i) The system is locally (Q, S, R)-dissipative in the region Ω only if there exists a storage function

$$\phi(x, t)\colon \Omega_c \times \boldsymbol{R} \to \boldsymbol{R}, \qquad 0 \le \phi(x, t) \le \infty, \qquad \forall x \in \Omega_c.$$

ii) The system is locally (Q, S, R)-dissipative in the region Ω_c if there exists a storage function

$$\phi(x, t)\colon \Omega_c \times \boldsymbol{R} \to \boldsymbol{R}, \qquad 0 \le \phi(x, t) \le \infty, \qquad \forall x \in \Omega_c.$$

Note that when $\Omega = \Omega_c$, i) and ii) can be combined and stated as: the system is locally (Q, S, R)-dissipative in a region Ω if, and only if, there exists a storage function

$$\phi(x, t)\colon \Omega \times \boldsymbol{R} \to \boldsymbol{R}, \qquad 0 \le \phi(x, t) \le \infty, \qquad \forall x \in \Omega.$$

Proof:

i) We will show that when the system is (Q, S, R)-dissipative, then the required supply $\phi_r(x_2, t_2)$, as defined by (2), is a storage function. Clearly, $0 \le \phi_r(x_2, t_2) < \infty$, $\forall x_2 \in \Omega_c$ and $\phi_r(0, t) = 0$, $\forall t \ge 0$. For any $x(t_0) = 0$, $x(t_1) = x_1 \in \Omega_c$, $x(t_2) = x_2 \in \Omega_c$ we know that $U(t_0, t_1, 0, x_1)$ and $U(t_1, t_2, x_1, x_2)$ are nonempty sets. For any $u_0 \in U(t_0, t_1, x_0, x_1)$ and $u_1 \in U(t_1, t_2, x_1, x_2)$ [clearly $u_0 \circ u_1 \in U(t_0, t_2, 0, x_2)$], we have

$$\phi_r(x_2, t_2) \le \int_{t_0}^{t_1} w(u_0, y)\, dt + \int_{t_1}^{t_2} w(u_1, y)\, dt. \tag{4}$$

Also, we know that

$$\phi_r(x_1, t_1) = \inf_{\substack{u \in U(t_0, t_1, 0, x_1) \\ t_1 \ge t_0,\, t_0 \in [-\infty, t_1]}} \int_{t_0}^{t_1} w(u, y)\, dt.$$

The infimum in the above case may not take place on a $u \in U(t_0, t_1, 0, x_1)$, but it is still true that given any $\epsilon > 0$, $\exists t_0 \in \boldsymbol{R}$, $u_m \in U(t_0, t_1, 0, x_1)$, such that

$$\int_{t_0}^{t_1} w(u_m, y)\, dt - \phi_r(x_0, t_0) < \epsilon.$$

Then (4) becomes $\forall u \in U(t_0, t_2, 0, x_2)$

$$\phi_r(x_2, t_2) \le \phi_r(x_0, t_0) + \epsilon + \int_{t_1}^{t_2} w(u_1, y)\, dt.$$

This is true $\forall \epsilon > 0$, giving the desired result

$$\phi_r(x_2, t_2) \le \phi_r(x_0, t_0) + \int_{t_1}^{t_2} w(u_1, y)\, dt.$$

ii) We have $\phi(0, t) = 0$, $\forall t \ge 0$, and from the definition of the storage function given by (1) we can write

$$\int_0^{t_1} w(u, y)\, dt \ge \phi(x_1, t_1) \ge 0, \qquad \forall u \in U(0, t_1, 0, x_1)$$

implying local (Q, S, R)-dissipativity in a region Ω_c.

Lemma 2: Let a dynamical system, locally connected in a region Ω_c wrt Ω, be locally (Q, S, R)-dissipative in the region Ω, and let $\phi(\cdot, \cdot)$ be one of its storage functions. Then

$$0 \le \phi_a(x_0, t_0) \le \phi(x_0, t_0) \le \phi_r(x_0, t_0), \qquad \forall x_0 \in \Omega_c.$$

Proof: For any storage function ϕ: $\Omega_c \times \boldsymbol{R} \to \boldsymbol{R}$, we have $\forall u \in U(t_0, t_1, x_0, x_1)$

$$\phi(x_0, t_0) + \int_{t_0}^{t_1} w(u, y)\, dt \ge \phi(x_1, t_1)$$

so when $x_0 = 0$, we get from above

$$\phi_r(x_1, t_1) \ge \phi(x_1, t_1). \tag{5}$$

Again, we have $\forall u \in U(t_0, t_1, x_0, x_1)$

$$\phi(x_0, t_0) \ge \phi(x_1, t_1) - \int_{t_0}^{t_1} w(u, y)\, dt$$

giving $\phi(x_0, t_0) \ge \phi(x_1, t_1) + \phi_a(x_0, t_0)$; and since $\phi(x_1, t_1) \ge 0$, we have

$$\phi_a(x_0, t_0) \le \phi(x_0, t_0). \tag{6}$$

Combining (5) and (6), we have

$$0 \le \phi_a(x_0, t_0) \le \phi(x_0, t_0) \le \phi_r(x_0, t_0), \qquad \forall x_0 \in \Omega_c.$$

Definition 4: A dynamical system is said to be locally uniformly controllable at $x_0 \in \text{int}(\Omega)$, if there exists an open neighborhood B_ϵ of x_0, such that for any $x_1 \in B_\epsilon$, there exist choices of $u \in U_e$ and t_1, such that the state can be driven from $x(t_0) = x_0$ to $x(t_1) = x_1$, $\{x(t) \in \Omega, t_0 \le t \le t_1\}$ with the additional property that

$$\left|\int_{t_0}^{t_1} w(u, y)\, dt\right| \leq \rho(\|x_1 - x_0\|)$$

for some continuous function $\rho: \boldsymbol{R}^+ \to \boldsymbol{R}^+$, such that $\rho(0) = 0$. The dynamical system is said to be locally uniformly controllable in Ω if it is locally uniformly controllable at every state $x_0 \in \operatorname{int}(\Omega)$.

Lemma 3: Let a dynamical system be locally uniformly controllable in a region $\Omega \subseteq X$. Then any storage function that exists $\forall x \in \Omega$ is also continuous.

Proof: Consider some arbitrary state $x_0 \in \operatorname{int}(\Omega)$ and let the storage function be $\phi(\cdot, \cdot)$. Then for any x_1 in the neighborhood B_ϵ of x_0, we have $\forall u \in U(t_0, t_1, x_0, x_1)$

$$\phi(x_0, t_0) + \int_{t_0}^{t_1} w(u, y)\, dt \leq \phi(x_1, t_1)$$

implying $|\phi(x_1, t_1) - \phi(x_0, t_0)| \leq \rho(\|x_1 - x_0\|)$. The arbitrary choice of x_1 and continuity of $\rho(\cdot)$ give that $\phi(\cdot)$ is continuous at $x_0, \forall x_0 \in \operatorname{int}(\Omega)$.

Definition 5: A dynamical system is said to be locally zero state detectable in a region Ω_z wrt Ω, if for any $x_0 \in \Omega_z$, $x_0 \neq 0$, such that $x(t) \in \Omega$, for $0 \leq t \leq \delta$, for some $\delta > 0$, with $u(\cdot) \equiv 0$, there exists a continuous function $\alpha: \boldsymbol{R} \to \boldsymbol{R}$ with $\alpha(0) = 0$ and $\alpha(\sigma) > 0$, $\forall \sigma > 0$, such that

$$\int_0^T y^T y\, dt \geq \alpha(|x_0|)$$

for some finite T such that $0 \leq T \leq \delta$. If, in addition, for any sequence $\{\sigma_n\} \in \Omega$, $\alpha(\sigma_n) \to \infty$ as $\|\sigma_n\| \to \infty$, the system is called locally uniformly zero state detectable in Ω_z wrt Ω.

Definition 6: A dynamical system is said to have a property A if there exists a well-defined feedback law $u^*(\cdot)$ such that $w(u^*(\cdot), y) < 0$, $\forall y \neq 0$, $u^*(0) = 0$ and $u^* \in U(t_0, t_1, x_0, x_1)$.

Lemma 4: Let the dynamical system be as follows:

i) Locally (Q, S, R)-dissipative in a region $\Omega \subseteq X$.
ii) Locally uniformly controllable in a region Ω_{uc} wrt Ω.
iii) Locally uniformly zero state detectable in region Ω_z wrt Ω.

Suppose, also, that the region $\Omega_s \triangleq \Omega_{uc} \cap \Omega_z$ is a nonempty region. Then the dynamical system has all its storage functions $\phi: \Omega_s \times \boldsymbol{R} \to \boldsymbol{R}$ continuous, $\phi(0) = 0$ and $\phi(x) > 0$, $\forall x \in \Omega_s$, $x \neq 0$. If, in addition, the system is locally uniformly zero state detectable in a region Ω_z wrt Ω, then for any sequence $\{x_n\} \in \Omega_s$, $\phi(x_n) \to \infty$ as $\|x_n\| \to \infty$.

Proof: Lemma 3 gives the continuity. Local zero state detectability gives $\phi_a(x_0) > 0$, $\forall x_0 \in \Omega_z$, $x_0 \neq 0$ and $\phi_a(0) = 0$. This, with the fact that

$$0 \leq \phi_a(x_0, t_0) \leq \phi(x_0, t_0) \leq \phi_r(x_0, t_0), \qquad \forall x_0 \in \Omega_c$$

gives the above result. From the definition of local uniform zero state detectability, it is clear that for any sequence $\{x_n\} \in \Omega$, $\phi(x_n) \to \infty$ as $\|x_n\| \to \infty$.

We have been fixing conditions on the dynamic system and the storage function so that $\phi(\cdot)$ can be used as a Lyapunov function candidate [3]. In the following claim, we show that the storage function $\phi(\cdot)$ can also be used to get a region of stability in addition to proving the local stability of the origin of the state space. We use compactness argument in the following claim to assure the existence of a minimum, so we assume that the metric $|\cdot|$ on the state space X is normable and $\|\cdot\|$ is the metric-induced norm. Let ∂A denote the boundary of a set A [4].

Claim 1: Let $\phi: \Omega \subset \boldsymbol{R}^n \to \boldsymbol{R}^+$ be continuous with the additional property that

i) $\phi(0) = 0$ and $\phi(x) > 0$, $\forall x \in \partial\Omega$.
ii) For any sequence $\{x_n\} \in \Omega_s$, $\phi(x_n) \to \infty$ as $\|x_n\| \to \infty$.

Then, there exists a subset $R \in \Omega$ defined by $\{x \in \Omega: \phi(x) \leq c\}$ such that $\phi(x) = c$, $\forall x \in \partial R$.

Proof: Choose any $x_0 \in \partial\Omega$, let $c_1 = \phi(x_0)$. Define $R_1 \triangleq \{x \in \Omega: \phi(x) \leq c_1\}$ and let $c = \min_{x \in \partial R_1}\{\phi(x)\}$. The minimum exists because ∂R_1 is compact and it is closed by definition. The set ∂R_1 is bounded, because if not, there would exist a sequence $\{x_n\} \in \partial R_1$ such that $\|x_n\| \to \infty$, implying that $\phi(x_n) \to \infty$ on ∂R_1, which is a contradiction. Moreover, $c > 0$, because $\phi(x) > 0$, $\forall x \in \partial\Omega$. Define $R_2 \triangleq \{x: \phi(x) \leq c\}$ and $R \triangleq R_1 \cap R_2$. Our claim is that $\phi(x) = c$, $\forall x \in \partial R$. Suppose the contrary is true, i.e., $\exists x_0 \in \partial R$ such that $\phi(x_0) \neq c$. By continuity of ϕ, $\phi(x_0) > c$ is impossible because x_0 is the limit of the sequence in R and $\phi(x) \leq c$, $\forall x$ in the sequence. Therefore $\phi(x) \leq c$. By continuity again, $\exists$ an open neighborhood of x_0 such that $\phi(x) \leq c$ in the neighborhood, which implies that $x_0 \notin \partial R_2$. Therefore $x_0 \in \partial R_1$, which is in contradiction with the definition of c.

For our next result, we need the following.

Local Existence and Uniqueness Theorem [3]: Let a system of differential equation be given by

$$\dot{x} = f(x, t), \qquad x \in \boldsymbol{R}^n, \qquad f: \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n.$$

Assume that the function f is continuous in t and x. Let B be a ball in $\boldsymbol{R}^n$ of the form, $B \triangleq \{x \in \boldsymbol{R}^n: \|x - x_0\| \leq r\}$. Assume that $\exists$ finite constants T, r, h, k such that

$$\|f(t, x) - f(t, y)\| \leq k\|x - y\|, \qquad \forall x, y \in B, \qquad \forall t \in [0, t] \tag{7}$$

$$\|f(t, x_0)\| \leq h, \forall t \in [0, t]. \tag{8}$$

Then, the system has exactly one solution over $[0, \delta]$ whenever $h\delta \exp(k\delta) \leq r$ and

$$\delta \leq \min\left(T, \frac{\rho}{k}, \frac{r}{h + kr}\right)$$

for some constant $\rho < 1$. Moreover

$$\|x(t) - x_0\| \leq h\delta \exp(kt) \leq r, \qquad \forall t \in [o, \delta].$$

IV. Main Results

First, we will fix conditions on the system and its local properties such that it is internally stable.

Theorem 1: Let the dynamical system

i) Be locally (Q, S, R)-dissipative in a region $\Omega \subseteq X$.
ii) Be locally connected in a region Ω_c wrt Ω.
iii) Be locally uniformly controllable in a region Ω_{uc} wrt Ω.
iv) Be locally uniformly zero state detectable in region Ω_z wrt Ω.
v) Be locally Lipschitz continuous in the region Ω.
vi) Have the property A.

Furthermore, suppose that the region $\Omega_s \triangleq \Omega_c \cap \Omega_{uc} \cap \Omega_z$ is nonempty, in the sense that it contains an open neighborhood of the origin. Then, if Q is negative definite, the origin is asymptotically stable.

Proof: Lemma 1 $\Rightarrow \exists \phi$: $\Omega_c \to \boldsymbol{R}$ such that, $\forall u \in U(t_0, t_1, x_0, x_1)$

$$\phi(x_0, t_0) + \int_{t_0}^{t_1} w(u, y)\, dt \leq \phi(x_1, t_1).$$

Lemma 3 $\Rightarrow \phi: \Omega_{uc} \to \boldsymbol{R}$ is continuous. Using this and the Claim 1, define $R \triangleq \{x \in \Omega_s: \phi(x) \le c\}$ and for some small $\epsilon > 0$, $R_\epsilon \triangleq \{x \in \Omega: \phi(x) \le c - \epsilon\}$, where c is as given in Claim 1.

Let $h(x_0)$ and $k(x_0)$ be the Lipschitz constants at x_0 as given in (7) and (8). Define $h^* \triangleq \max_{x_0 \in R_\epsilon}\{h(x_0)\}$ and $k^* \triangleq \max_{x_0 \in R_\epsilon}\{k(x_0)\}$. R_ϵ is compact, so the maxima exist and are finite by definition. Because h^*, k^* are finite, we have that there exists a finite δ^* such that $h^*\delta^* \exp(k^*\delta^*) = \epsilon$, where clearly $\delta^* > 0$. This, and the local existence and uniqueness theorem gives that $\|x(t) - x_0\| \le \epsilon$, $\forall t \in [0, \delta^*]$. This means that $\forall x_0 \in R_\epsilon$, $x(t) \in R$, $\forall t \in [0, \delta^*]$. So we have, when $u \equiv 0$

$$\phi(x(t)) - \phi(x(t_0)) \le \int_{t_0}^{t} y^t Q y \, dt, \; t_0 \le t \le t_0 + \delta^*$$

Q is negative definite giving $\phi(x(t)) \le \phi(x(t_0))$, $t_0 \le t \le t_0 + \delta^*$, we can repeat the above argument again with $x_0 = x(t_0 + \delta^*)$, so we have $\phi(x(t)) \le \phi(x(t_0))$, $\forall t \ge t_0$. Because of the local uniform zero state detectability, we have $\phi(x(t_1)) \le \phi(x(t_0))$, for some $t_0, t_1 > 0$, whenever $x(t_0) \ne 0$. Lower bound is given by local uniform controllability assumption. Clearly $\phi(\cdot)$ is a Lyapunov function for the system.

Remark 1: The proof proceeds more or less on the standard lines as in [6]. The difference is only in the fact that here we need only a local description as opposed to the global needed elsewhere.

Remark 2: We need Lipschitz continuity arguments to assure that the system trajectory, if started in a subset of Ω, remains in Ω for a finite time, irrespective of the system stability. A proof of this statement can be found in [3]. This helps to straighten the circular argument in [5].

Remark 3: In proving the above theorem, we have also given an estimate of the region of stability R.

The next theorem refers to a linear interconnection of N locally dissipative subsystems. The interconnection is described by

$$u_i = u_{ei} - \sum_{j=1}^{N} H_{ij} y_j, \qquad i = 1, \cdots, n \tag{9}$$

where u_i is the input to subsystem i; y_i is the output; u_{ei} is an external input; and H_{ij} are constant matrices. A compact matrix notation is, with obvious definitions

$$U = U_e - Hy.$$

Theorem 2: Let the dynamical system be formed by interconnecting N subsystems via the interconnection (9) and suppose that:

i) The ith subsystem is locally (Q_i, S_i, R_i)-dissipative in a region Ω_i satisfying conditions i)–vi) of Theorem 1.

ii) The interconnection (9) is such that the dynamic system state space $\hat{X}$ equal to the Cartesian product of the state space of the individual subsystems and $\Omega = \Omega_1 \times \cdots \times \Omega_N$ is a nonempty region containing a neighborhood of the origin ($\Omega \subseteq \hat{X}$).

iii) The overall system is uniformly zero state detectable in a region Ω_z wrt Ω, where Ω_z is a nonempty region containing a neighborhood of the origin ($\Omega_z \subseteq \hat{X}$).

iv) The overall system is locally Lipschitz continuous in Ω.

Then if $\hat{Q}$ is positive definite, the origin is asymptotically stable. Where

$$\hat{Q} = SH + H^T S^T - H^T R H - Q$$

$$Q = \{Q_1, \cdots, Q_N\}, S = \{S_1, \cdots, S_N\}, R = \{R_1, \cdots, R_N\}.$$

Proof: For each subsystem we have, by Lemma 1, $\forall u_i \in U_i(t_0, t_1, x_i(t_0), x_i(t_i))$

$$\phi_i[x_i(t_0)] + \int_{t_0}^{t_1} w_i(u_i, y_i) \, dt \ge \phi_i[x_i(t_i)], \qquad i = 1, \cdots, N, \qquad \forall t_1 \ge t_0.$$

Now let

$$V(x) = \sum_{i=1}^{N} \phi_i(x_i).$$

By Lemma 4 and the local uniform zero state detectability of each subsystem, we have $V(x) > 0$, $\forall x \in \Omega$, $x \ne 0$, $V(0) = 0$, and for any sequence $\{x_n\} \in \Omega$, $V(x_n) \to \infty$ as $\|x_n\| \to \infty$. Define $R_\epsilon \triangleq \{x \in \Omega: V(x) \le c - \epsilon\}$, where c is obtained by applying Claim 1 on $V(x)$. R_ϵ is a compact set. Repeating the arguments in Theorem 1 based on Lipschitz continuity, it can be shown that $V(x)$ is a Lyapunov function as per the definition in [6], such that

$$V[x(t_0)] + \int_{t_0}^{t_1} y^T Q y \, dt \ge V[x(t_i)], \forall t_1 \ge t_0$$

provided $x(t_0) \in R_\epsilon$.

Remark 4: The above two theorems, when put together, give a method of constructing Lyapunov functions for an interconnected system. Following is a step-by-step method to get a Lyapunov function.

Step 1) Decompose a large-scale system into N subsystems.

Step 2) Find out the *storage function* $\phi_i(x_i)$, and the corresponding (Q_i, S_i, R_i) for each subsystem, using methods discussed in [7] and [10].

Step 3) Form the matrix $\hat{Q} = SH + H^T S^T - H^T R H - Q$ and check for its positive definiteness. If $\hat{Q}$ is positive definite then

$$V(x) = \sum_{i=1}^{N} \phi_i(x_i),$$

is a Lyapunov function. If $\hat{Q}$ is not positive definite, then it becomes necessary to try another decomposition or a different choice of (Q_i, S_i, R_i).

The above three steps are successfully used to construct Lyapunov functions for large power systems in [10]–[12].

IV. Conclusion

The main result stated as Theorem 2 is just the type of result we need to study the behavior of an interconnected system whose subsystems are only "locally" defined. To obtain the result, we have made precise what is "locally" required of the subsystems. If the definitions are read carefully, one can see that "local" is both in term of small gain inputs [5] and local internal stability regions. Hence, this note can be considered as a contribution in extending the already-available results of [1], [5], and [8]. This extension has an immediate application for transient stability analysis of power systems [10]–[12].

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Lower Bounds for the Trace of the Solution of the Discrete Algebraic Riccati Equation

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***Abstract*—This note suggests two lower bounds for the trace of the solution of the discrete algebraic Riccati equation (DARE). It is shown that in many cases, the trace bounds of this note are tighter than those in the literature and greater than the trace of the state weighting matrix even when the system matrix is singular. The results are illustrated through an example.**

I. Introduction

Riccati and Lyapunov equations play a fundamental role in various areas of engineering problems such as control or filtering problems, in which it is often necessary to solve Lyapunov or Riccati equations. Since the computation of the solution causes some difficulty when the dimensions of the matrices involved are large, it is very helpful to obtain several handy bounds such as bounds for the trace, the eigenvalues (especially the minimum and maximum ones), the determinant, etc. During the last two decades, much research was carried out to find these bounds for the solutions of the discrete and continuous Riccati and Lyapunov equations [1]. Among them, the trace bounds are important since they give a "mean size" of the solution, i.e., $\mathrm{tr}(P)/n$ is the arithmetic mean of the eigenvalues of P, where $\mathrm{tr}(P)$ denotes the trace of an n-dimensional square matrix P. Some additional results on the trace bounds for the solution of the discrete algebraic Riccati equation have been recently reported in [1]–[5]. The lower bounds of the trace of Garloff [2] and Mori *et al.* [3] are trivially the trace of the state weighting matrix Q when the system matrix A is singular. This note is concerned with nontrivial trace bounds even when the system matrix A is singular. It is shown that the trace bounds of this note are always tighter than those of Garloff [2]. It is also shown that these bounds are tighter than those of Mori *et al.* [3] when the system matrix A is singular.

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This note is organized as follows. In Section II, several notations and preliminary facts are introduced. The main results are presented in Section III, and comparisons of the trace bounds of this note with those of Garloff [2] and Mori *et al.* [3] are made in Section IV. Finally, conclusions are in Section V.

II. Preliminaries

We define several notations as follows:

- $R^{n\times r}$ the set of real $n \times r$ matrices,
- I_r the identity matrix of order r,
- $\mathrm{tr}(X)$ the trace of a matrix $X \in R^{n\times n}$, and
- $\lambda_i(X)$ the ith eigenvalue of a matrix $X \in R^{n\times n}$.

The eigenvalues of $X \in R^{n\times n}$ are assumed to be arranged in decreasing order, i.e.,

$$|\lambda_1(X)| \geq |\lambda_2(X)| \geq \cdots \geq |\lambda_n(X)|.$$

Now, we consider the discrete algebraic Riccati equation (DARE) such as:

$$A^TPA - P - A^TPB(I_r + B^TPB)^{-1}B^TPA + Q = 0 \quad (2.1)$$

where A, P, and $Q \in R^{n\times n}$, $B \in R^{n\times r}$; Q is positive semidefinite; $(A, B, Q^{1/2})$ is minimal; and the superscript T means transposition. A lower bound for the minimum eigenvalue of the solution P of DARE (2.1) is known to be as follows [4]:

$$\lambda_n(P) \geq M \quad (2.2)$$

where M is defined as

$$M \triangleq \frac{2\lambda_n(Q)}{\sqrt{K^2 + 4\lambda_1(BB^T)\lambda_n(Q)} + K} \quad (2.3)$$

and K is given by

$$K \triangleq 1 - \lambda_1(BB^T)\lambda_n(Q) - \lambda_n(A^TA). \quad (2.4)$$

It is noted that the lower bound M of the minimum eigenvalue of the solution P is zero if the minimum eigenvalue of the matrix Q is zero. The above facts will be used to prove main results.

III. Main Results

In this section, we consider DARE (2.1). It is well known that, under the assumptions made in Section II, DARE (2.1) always has a unique solution P, which is symmetric and positive definite. The following theorem gives a lower bound on the trace of the solution P of DARE (2.1).

Theorem 3.1: The trace of the solution P of DARE (2.1) has the following relation:

$$\mathrm{tr}(P) \geq b_1 \quad (3.1)$$