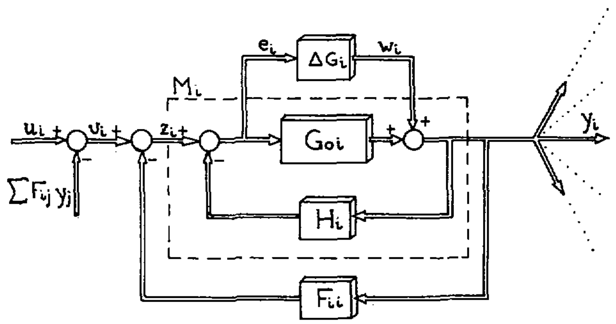

Fig. 2. The ith subsystem S_i for a particular internal structure of M_i.

$$C_1=(I+GH)^{-1},\quad C_2=(I+GH)^{-1}G \tag{15}$$

$$E_1=H(I+GH)^{-1},\quad E_2=(I+HG)^{-1} \tag{16}$$

$$G=\mathrm{diag}(G_i),\quad H=\mathrm{diag}(H_i). \tag{17}$$

By substitution of (15) and (16) into (12), after some manipulations we obtain

$$Q=(I+(G+\Delta G)H)^{-1}(G+\Delta G). \tag{18}$$

Let us now introduce a class $\Delta\mathcal{G}$ of linear diagonal operators ΔG_λ which depend on a parameter λ belonging to a parameter set Λ. Letting ΔG_λ replace ΔG in (18), we define the diagonal operator

$$Q_\lambda=(I+(G+\Delta G_\lambda)H)^{-1}(G+\Delta G_\lambda). \tag{19}$$

Finally we introduce a $m\times m$ matrix $R_\lambda=(r_{ij}(\lambda))$ whose elements $r_{ij}(\lambda)$ are given by

$$r_{ij}(\lambda)=f_{ij}\|Q_{j\lambda}\| \tag{20}$$

where $Q_{j\lambda}$ is the jth suboperator of Q_λ.

On the basis of the previous considerations the following theorem holds.

Theorem 2: Assume that $\Delta G=\Delta G_{\hat\lambda}$ for a suitable value $\hat\lambda\in\Lambda$. Assume, moreover, that the following conditions hold:

a) C_1, C_2, E_1, E_2, F are bounded causal operators;

b) Q_λ is a bounded causal operator for all $\lambda\in\Lambda$;

c) $\rho(R_{\lambda^*})<1$ where $\|Q_{i\lambda^*}\|=\sup_{\lambda\in\Lambda}\|Q_{i\lambda}\|$, $i=1,\cdots,m$.

Then the system given by (6)–(9) and (15)–(17) is input–output stable.

Proof: From Theorem 2.b) we note that Q is bounded since $\Delta G=\Delta G_{\hat\lambda}$ and $\hat\lambda\in\Lambda$. Moreover, from Theorem 2.a) and 2.b) we deduce that for every $i=1,\cdots,m$ there exists a pair of constants α_i and β_i ($\beta_i\geqslant 1$) such that

$$\|Q_i\|=\alpha_i\|C_{2i}\| \tag{21}$$

$$\|Q_{i\lambda^*}\|=\beta_i\|Q_i\| \tag{22}$$

where λ^* corresponds to the maximum value $\|Q_{i\lambda^*}\|$ of the gain $\|Q_{i\lambda}\|$. Consider now condition c) of Theorem 2, then from (13), (20)–(22) it can be verified that

$$\rho(R)=\rho(R_{\hat\lambda})\leqslant\rho(R_{\lambda^*})<1.$$

This proves the theorem since the proof of Theorem 1 is based on Assumption a) of Theorem 2, $Q_{\hat\lambda}$ is bounded and causal, and $\rho(R_{\hat\lambda})<1$. ■

By comparison of the above theorem with the stability criterion [7] we deduce that Theorem 2 allows the following two level analysis:

1) at subsystem level, by letting λ run over the uncertainty range, we evaluate $\|Q_{i\lambda^*}\|$;

2) at system level, R_{λ^*} is constructed and the above condition c) of Theorem 2, is tested;

and it can also be applied to cases where ΔG is unstable. Consider now subsystem S_i (Fig. 2): C_{2i}, $Q_{i\hat\lambda}$, $Q_{i\lambda^*}$ as given by (15) and (19) represent input–output operators of, respectively, the unperturbed, perturbed, and worst case subsystems; (21) and (22) are norm relations for the perturbed–unperturbed system and worst case-perturbed system comparison. These relations are convenient in the design stage: we can see how the perturbed subsystem deviates from the unperturbed system (α_i) and how the worst case relates to the unperturbed case ($\|Q_{i\lambda^*}\|=\|C_{2i}\|\alpha_i\beta_i$). Notice that in the proof of Theorem 2 above, the stability of the worst case overall-model ensures the stability of all perturbed models with $\hat\lambda\in\Lambda$: if $\Delta G_{\hat\lambda}=0$ for some $\hat\lambda\in\Lambda$, then the stability of the unperturbed case is also ensured. Since, moreover, only the products $\alpha_i\cdot\beta_i$ of the constants appearing in (21) and (22) need to be evaluated, the criterion presented here still retains the mean feature, expressed in [7], of using reduced order models for the design of feedback stabilized subsystems.

V. Conclusions

Modeling is undoubtely the most important step of any analysis and synthesis procedure. When large-scale system stability problems are considered, the effect of modeling errors at subsystem level must be carefully investigated. The results of this correspondence prove that it is possible to deduce guaranteed stability margins in the case of particular types of model uncertainties as represented in Figs. 1 and 2. Moreover, it has been shown that, once the subsystem input–output properties have been analyzed by using their simplified models, the well-known composite-system method can be applied to test the stability of the whole system.

References

[1] N. R. Sandell, Jr., P. Varaiya, M. Athans, and M. G. Safonov, "Survey of decentralized control methods for large scale systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 108–128, Apr. 1978.

[2] N. H. McClamrock, "A representation for multivariable feedback systems and its use in stability analysis. Part I: Linear systems, "*Int. J. Contr.*, vol. 24, pp. 83–95, 1976; "Part II: Non-linear systems," *Int. J. Contr.*, vol. 24, pp. 97–107, 1976.

[3] M. Araki, "Input–output stability of composite feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 254–259, Apr. 1976.

[4] ——, , "Stability of large-scale nonlinear systems—Quadratic-order theory of composite-system method using *M*-matrices," *IEEE Trans. Automat. Contr.* vol. AC-23, pp. 129–142, Apr. 1978.

[5] P. J. Moylan and D. J. Hill, "Stability criteria for large-scale systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 143–149, Apr. 1978.

[6] F. M. Callier, W. S. Chan, and C. A. Desoer, "Input–output stability of interconnected systems using decompositions: An improved formulation," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 150–163, Apr. 1978.

[7] R. A. El-Attar and M. Vidyasagar, "Subsystem simplification in large-scale systems analysis," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 321–323, Apr. 1979.

[8] P. J. Moylan and D. J. Hill, "Tests for stability and instability of interconnected systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 574–575, Aug. 1979.

[9] M. Vidyasagar, "New passivity-type criteria for large-scale interconnected systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 575–579, Aug. 1979.

[10] I. W. Sandberg, "Some results on the theory of physical systems governed by non-linear functional equations," *Bell Syst. Tech. J.*, vol. 44, pp. 871–898, 1965.

[11] G. Zames, "On the input—output stability of time-varying nonlinear feedback systems, part I: Conditions derived using concepts of loop gain, conicity and positivity," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 228–238, Apr. 1966.

[12] J. C. Willems, "Stability, instability, invertibility and causality," *SIAM J. Contr.*, vol. 7, no. 4, pp. 645–671, 1969.

[13] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input–Output Properties*. New York: Academic, 1975.

[14] D. Carlucci and F. Donati, "Control of norm uncertain systems," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 792–795, Dec. 1975.

[15] N. R. Sandell, Jr., "Robust stability of systems with application to singular perturbations," *Automatica*, vol. 15, pp. 467–470, 1979.

[16] M. G. Safonov and M. Athans, "A multiloop generalization of the circle criterion for stability margin analysis," Massachusetts Inst. Tech., Cambridge, Rep. LIDS-P-946, 1979.

A Relaxed Dissipativeness Test for the Stability of Large-Scale Systems

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***Abstract*—A recently published stability criterion allows checking the stability of a large-scale system whose subsystems are dissipative. (Dissipativeness is a property akin to, but more general than, passivity.) The criterion requires that a certain test matrix be positive definite. It is now**

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shown that one can sometimes deduce stability when the test matrix is merely nonnegative definite.

I. Introduction

In a recent paper [1], a general stability criterion was given for a wide class of interconnected systems. Each subsystem was assumed to be "dissipative"—a property which includes passivity, finite gain, and conicity as special cases—but was otherwise largely unrestricted: it could be linear or nonlinear and be memoryless or have lumped or distributed memory. Both Lyapunov stability and input–output stability were treated. The interconnections between the subsystems were assumed to be linear, with all nonlinearities absorbed into the subsystems.

A central result of [1] is that stability can be checked by forming a test matrix $\hat{Q}$. A sufficient condition for stability is that $\hat{Q}$ be positive definite. If the test matrix $\hat{Q}$ is nonnegative definite, but singular, then intuitively one would suppose that stability could still be guaranteed by adding some minor side constraints. For example, the knowledge that some of the subsystems were stable in isolation, and that these subsystems were "sufficiently well coupled" into the overall system, might be sufficient. The present note shows that one can, in fact, simply state a general stability criterion which is applicable in the case where the test matrix $\hat{Q}$ of [1] is nonnegative definite.

The results presented here have several points of contact with those of Vidysagar [3]. In an earlier paper [4], Vidyasagar gave a passivity-type stability criterion; as in [1], this required that a certain test matrix be positive definite. The results of [3] are intended to apply to those situations where it is difficult to check whether the test matrix is positive definite. Both sets of results are based on adding a rank condition to a test for nonnegative definiteness. (Theorem 2 of this paper actually implies the main stability result of [3], but the proof of this assertion is sufficiently long that it is more natural to treat the present results and those of [3] as independent criteria.) The notion of using a rank test to ensure that certain of the subsystems are "sufficiently well coupled" to the remainder was also exploited in the earlier paper [5].

II. Dissipative Systems

Let u_i denote the input and y_i the output of the ith subsystem. Each subsystem may, in general, be nonlinear and/or infinite dimensional, and may have multiple inputs and outputs. Let P_T be the operator which truncates a signal at time T. It is assumed that, for all $T<\infty$, $P_T u_i$ and $P_T y_i$ belong to some suitable inner product space. The notation $\langle u, v\rangle_T$ means $\langle P_T u, P_T v\rangle$.

Subsystem i is called (Q_i, S_i, R_i)-dissipative for matrices Q_i, S_i, and R_i where Q_i and R_i are self-adjoint if the inequality

$$\langle y_i, Q_i y_i\rangle_T + 2\langle y_i, S_i u_i\rangle_T + \langle u_i, R_i u_i\rangle_T \geqslant 0$$

is satisfied for all u_i and all $T<\infty$.

Now let

$$u = u_{\text{ext}} - Hy$$

where u and y are vectors formed by concatenating all the u_i and y_i, respectively, u_{ext} is an external input, and H is a constant matrix. That is, the overall system is a linear interconnection of the possibly nonlinear subsystems. The overall system has input u_{ext} and output y.

Assume that all subsystems are dissipative in the above sense where the Q_i, S_i, and R_i may be different for each i. Let $Q=\text{diag}\{Q_1, Q_2, \cdots\}$, $S=\text{diag}\{S_1, S_2, \cdots\}$, and $R=\text{diag}\{R_1, R_2, \cdots\}$. Form the matrix

$$\hat{Q} = SH + H^T S^T - H^T R H - Q. \tag{1}$$

It was shown in [1] that positive definiteness of $\hat{Q}$ is a sufficient condition for input–output stability. With minor additional technical assumptions, it is also sufficient for Lyapunov stability. It was also shown in [1] that, for various choices of the Q_i, S_i, and R_i, the condition $\hat{Q}>0$ implies many of the known stability criteria for interconnected systems.

If $\hat{Q}$ is merely positive semidefinite, stability is no longer guaranteed. It will be shown, however, that if $\hat{Q}\geqslant 0$ *and* if some side conditions are imposed—for example, if one has the additional information that some of the subsystems have finite gain—then stability may still be deduced.

III. A General Stability Criterion

Suppose now that each subsystem is dissipative with respect to two (Q, S, R) triples. For example, some of the subsystems might be known to be both passive and finite gain. More specifically, suppose that, for each i, subsystem i is both $(Q_i^{(1)}, S_i^{(1)}, R_i^{(1)})$-dissipative and $(Q_i^{(2)}, S_i^{(2)}, R_i^{(2)})$-dissipative. [This formulation is less restrictive than it might appear to be because for each i such that a second (Q_i, S_i, R_i) cannot be found, we can always set $(Q_i^{(2)}, S_i^{(2)}, R_i^{(2)})=(0,0,0)$.] Under these conditions, it is clear that subsystem i is also $(Q_i^{(1)}+\alpha Q_i^{(2)}, S_i^{(1)}+\alpha S_i^{(2)}, R_i^{(1)}-\alpha R_i^{(2)})$-dissipative for any real positive (or zero) α.

As in Section II, let $Q=\text{diag}\{Q_1^{(1)}+\alpha Q_1^{(2)}, Q_2^{(1)}+\alpha Q_2^{(2)}, \cdots\}$, and similarly for S and R. Then the matrix $\hat{Q}$ defined in (1) has the form

$$\hat{Q} = \hat{Q}^{(1)} + \alpha \hat{Q}^{(2)} \tag{2}$$

where $\hat{Q}^{(1)}$ is the matrix which would result from (1) if only the $(\hat{Q}_i^{(1)}, \hat{S}_i^{(1)}, \hat{R}_i^{(1)})$ were taken into consideration, and $\hat{Q}^{(2)}$ similarly arises from the quantities with "2" superscripts.

By setting $\alpha=0$, we revert to the main result of [1]: if $\hat{Q}^{(1)}$ is positive definite, the interconnected system is stable. If $\hat{Q}^{(1)}$ is merely nonnegative definite, stability can still be guaranteed if there exists any $\alpha>0$ such that $\hat{Q}$ in (2) is positive definite.

More typically, $\hat{Q}^{(2)}$ will have no special sign properties, but we can always write $\hat{Q}^{(2)}=Q_B-Q_C$ where Q_B and Q_C are both nonnegative definite.

Lemma: Let Q_A, Q_B, Q_C be three symmetric nonnegative definite $n\times n$ matrices such that

$$\text{rank}[Q_A \quad Q_B]=n$$

$$\text{rank}[Q_A \quad Q_C]=\text{rank}[Q_A].$$

Then there exists a real $\alpha>$ such that $Q_A+\alpha(Q_B-Q_C)$ is positive definite.

Proof: If Q_A is the zero matrix, the proof is trivial. Otherwise, choose any α such that

$$0<\alpha<\frac{\lambda_{\text{snz}}(Q_A)}{\lambda_{\max}(Q_C)}$$

where $\lambda_{\max}(Q_C)$ is the largest eigenvalue of Q_C and $\lambda_{\text{snz}}(Q_A)$ is the smallest *nonzero* eigenvalue of Q_A. It is then straightforward to show, from the second rank condition, that $Q_A-\alpha Q_C$ is nonnegative definite. (This result is Lemma 2 in [3].) Then for any vector y we have

$$y^T(Q_A-\alpha Q_C)y\geqslant 0$$

and

$$y^T(\alpha Q_B)y\geqslant 0.$$

If y is nonzero, the above two rank conditions imply that these quantities cannot be simultaneously zero, and our result now follows easily. ▽▽▽

Theorem 1: Let $\hat{Q}^{(2)}=Q_B-Q_C$ where both Q_B and Q_C are nonnegative definite. Then a sufficient condition for stability of the interconnected system is that the three conditions

$$\hat{Q}^{(1)}\geqslant 0$$

$$\text{rank}\left[\hat{Q}^{(1)} \quad Q_B\right]=n$$

$$\text{rank}\left[\hat{Q}^{(1)} \quad Q_C\right]=\text{rank}\left[\hat{Q}^{(1)}\right]$$

hold where n is the total number of outputs of all the systems. ▽▽▽

Proof: The proof is obvious from the lemma. ▽▽▽

IV. Example

Suppose we have three passive systems and an interconnection matrix

$$H=\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Passivity corresponds to $Q^{(1)}=0$, $R^{(1)}=0$, and $S^{(1)}=I$ (the unit matrix). After substitution in (1), we have

$$\hat{Q}^{(1)}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix is nonnegative definite, but singular, so we cannot yet conclude that the interconnected system is stable.

Suppose, however, that the second subsystem satisfies the constraint

$$\langle y_2, y_2 \rangle_T \geqslant \epsilon \langle u_2, u_2 \rangle_T$$

for some $\epsilon>0$. That is, it is $(1/\epsilon, 0, -1)$-dissipative. This allows us to write, from (1),

$$\hat{Q}^{(2)}=S^{(2)}H+H^TS^{(2)T}-H^TR^{(2)}H-Q^{(2)}$$

where $S^{(2)}$ is zero, $R^{(2)}=\text{diag}(0,-1,0)$, and $Q^{(2)}=\text{diag}(0,1/\epsilon,0)$. The result of this calculation is

$$\hat{Q}^{(2)}=\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}-\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\epsilon} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the conditions of Theorem 1 are satisfied, so we may deduce that the system is stable.

V. Finite Gain Constraints

An important special case arises when it is known that some of the subsystems have finite gain. That is, when it is known that subsystem i is (Q_i, S_i, R_i)-dissipative (for all i) and that $\|y_i\|_T \leqslant k_i \|u_i\|_T$ (for some but possibly not all i) where the k_i are finite, but probably unknown real constants. We assume, as before, that the (Q_i, S_i, R_i) triples are used to form a matrix $\hat{Q}^{(1)}$ via (1), and that $\hat{Q}^{(1)}$ turns out to be nonnegative definite and singular.

Let K be a diagonal matrix formed from the k_i, with zero entries corresponding to those subsystems which are not known to have the finite gain property. Also let D be a diagonal matrix, with unit entries corresponding to those outputs with the "finite gain" property and zeros elsewhere. Note that this formulation allows for the possibility of subsystems with multiple outputs, some of which have the finite gain property and some which do not. Forming $\hat{Q}^{(2)}$ as in Section III, the result is

$$\hat{Q}^{(2)}=D-H^TKDKH.$$

This leads to the following result.

Theorem 2: With all matrices as defined above, sufficient conditions for stability are

$$\hat{Q}^{(1)} \geqslant 0$$

$$\text{rank}[\hat{Q}^{(1)} \quad D]=n$$

$$\text{rank}\begin{bmatrix} Q^{(1)} \\ DH \end{bmatrix}=\text{rank}[\hat{Q}^{(1)}]$$

where n is the total number of outputs.

Proof: From Theorem 1, sufficient conditions for stability are $\hat{Q}^{(1)} \geqslant 0$ and

$$\text{rank}[\hat{Q} \quad D]=n$$

$$\text{rank}[\hat{Q}^{(1)} \quad H^TKDKH]=\text{rank}[\hat{Q}^{(1)}].$$

It is a simple exercise to show that these conditions are, because D and K are diagonal, equivalent to the conditions stated above. ▽▽▽

Notice that K does not appear in the theorem statement. To apply this result, we only have to know which subsystems have finite gain. It is not necessary to know the gain bounds.

If we apply this result to the example of Section IV, but without the side constraint on the second subsystem, it is easy to see that a sufficient condition for stability is $D=\text{diag}(0,0,1)$. The requirement is therefore that subsystem 3 have finite gain. In summary, the system of this example is stable if the second subsystem has a lower bound on its gain *or* if the third subsystem has an upper bound on its gain.

References

[1] P. J. Moylan and D. J. Hill, "Stability criteria for large-scale systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 143–149, Apr. 1978.

[2] D. Šiljak, *Large-Scale Dynamic Systems*. Amsterdam: North-Holland, 1978.

[3] M. Vidyasagar, "New passivity-type criteria for large-scale interconnected systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 575–579, Aug. 1979.

[4] ——, "L_2-stability of interconnected systems using a reformulation of the passivity theorem," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 637–645, 1977.

[5] M. K. Sundareshan and M. Vidyasagar, "L_2-stability of large-scale dynamical systems: Criteria via positive operator theory," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 396–399, June 1977.

Comments on Decentralized State Feedback Stabilization

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Abstract—**The problem of decentralized stabilization of interconnected systems using state feedback is revisited. Various aspects of the problem are briefly discussed with examples. A conjecture, which is believed to provide sufficient conditions for decentralized stabilizability of interconnected systems is proposed along with the supporting reasons.**

Recently a discussion was started by Wang [1] about stabilization of interconnected systems composed of the subsystems

$$\dot{x}_i=A_ix_i+\sum_{\substack{j=1\\ j\neq i}}^{N} A_{ij}x_j+B_iu_i, \qquad i\in N \tag{1}$$

using decentralized state feedback

$$u_i=K_ix_i, \qquad i\in N \tag{2}$$

where $N=\{1,2,\cdots,N\}$. Wang gave an example of a system consisting of $N=3$ subsystems which cannot be stabilized by decentralized feedback although each subsystem as well as the overall system is controllable, thus showing that the corresponding result by Aoki and Li [2] for the case $N=2$ does not readily extend to the case $N\geqslant3$.

We first would like to point out that the very intuitive proof suggested by Aoki and Li is, unfortunately, not correct. They suggest, as a stabilization procedure, to shift the subsystem eigenvalues by decentralized feedback to locations far away from the imaginary axis in the left-half complex plane, so that the resulting closed-loop system becomes weakly coupled [3], and, hence, stable. The critical point in this argument is

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