Dissipative Dynamical Systems: Basic Input–Output and State Properties†

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ABSTRACT: *A complete account is given of the theory of so-called dissipative dynamical systems. The concept of dissipativeness is defined as a general input–output property which includes, as notable special cases, passivity and other properties related to finite-gain. The aim is to treat input–output and state properties side-by-side with emphasis on exploring connections between them. The key connection is that a dissipative system in general possesses a set of energy-like functions of the state. The properties of these functions are studied in some detail. It is demonstrated that this connection represents a direct generalization of the well-known Kalman–Yakubovich lemma to arbitrary dynamical systems. Applications to stability theory and passive system synthesis are briefly discussed for non-linear systems.*

I. Introduction

The mathematical representation of physical systems generally takes one of two forms: an input–output description, where the system is regarded as a mapping of input functions to output functions; or a state–space description, which describes the system in terms of trajectories in a metric space (or flows on appropriate manifolds). These choices of representation are exemplified by the familar convolution (or transfer function) and differential equation representations of linear systems. The role of the relationship between these representations and the evolution of system theory (particularly control and network theory) is, of course, well-known. It is now accepted that each approach complements the other to provide a firm foundation for system theory. Theoretically, the input–output description provides the benefits of abstraction; because it is free of details about the internal description, basic results in system theory can be viewed more easily. In system design, this approach facilitates designing for a prescribed response to a specified class of inputs. When internal constraints are to be accounted for, the extra information concerning the state–space is needed. Studies in this setting facilitate designing for prescribed internal system modes and qualitative behaviour of trajectories. Evidently, the foundations of system theory must include a study of the relationships between the two descriptions of a system. There is a

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massive literature on this topic for linear systems with many of the basic results for the finite-dimensional case given in (**1**).

Such a theory for general non-linear systems does not exist and would not appear to be quickly forthcoming. However, progress is being made by restricting the class of non-linear systems and attempting to generalize as many of the linear systems results as possible (**2, 3**). Certainly such an approach is essential for obtaining algebraic or graphical tests for system structural properties. On the other hand, if we confine our attention to the study of qualitative properties, to a large extent the theoretical developments can proceed without restrictive *a priori* structural constraints. This is illustrated by the area of stability theory (**4–6**), where the main theorems are derived for general dynamical system representations. Taking this approach, of course, puts computational aspects aside; at least until more system structure is imposed.

It is the intention of this paper to study a general theory for relating input–output and internal system qualitative behaviour. We assume that the systems satisfy a general time–domain inequality called dissipativeness (which in the memoryless system case corresponds to its graph being confined to lie in a sector). It is then possible to give, in a very explicit manner, the implications of this input–output property on a state–space representation.

The study of so-called dissipative systems was initiated by Willems (**7**) in order to tie together ideas common to network theory and feedback control theory, as well as thermodynamics and mechanics. This work can be seen as evolving from a series of studies beginning with the well-known Kalman–Yakubovich lemma (**8, 9**) and its applications; see, for instance, Refs. (**10–13**). These studies can be interpreted as exploring the usefulness of the concept of passivity (or positive real transfer functions).

The essence of dissipative system theory is the broadening, in a theoretical sense, of the meaning of energy storage. The property of dissipativeness was defined in (**7**) essentially as a generalization of the property of passivity via an inequality based on a state–space description. Associated with a dissipative system is an energy-like framework akin to that occurring for physical systems; except in that, in general, the stored energy functions are non-unique. In a subsequent publication (**14**), Willems introduced a weaker property which he called cyclo-dissipativeness, which amounts to dissipativeness on cyclic motions. For linear systems, it was shown that these concepts unify many important system properties such as stability, reciprocity and reversibility with energy storage ideas (**7**). In summary, dissipativeness was introduced as a property which reflects something of the internal properties of the system.

The present authors have been interested in dissipative systems primarily as a vehicle for producing very general stability results for interconnected systems (**15–19**). In the process of carrying out this work extensions to the theory in (**7, 14**) were made, i.e. consideration of dissipative systems in a purely operator theoretic setting, clarifying the role of minimality of the state–space representation, and providing algebraic tests for dissipativeness of classes of non-linear systems. Some of these extensions appeared in (**20**), whereas others are scattered throughout the various papers on stability. Much of this work was carrying over known results for special cases to the more general situation to provide as general a framework as possible for applications. It should also be noted that some extensions to the theory in (**7**, **14**) were also provided in (**21**). The purpose of this paper is to collect together and further extend the essential features of the theory of dissipative systems. Some of the results are only variations of those given by Willems (**7**, **14**, **21**), but the overall intention is to provide a complete background for applications of the theory.

The structure of the paper is as follows. Section II defines dissipativeness with various refinements (including an input–output viewpoint of cyclo-dissipativeness) in an operator theoretic setting. A known result relating to causality and passive operators (**22**) is generalized. Sections III and IV are based on Ref. (**20**): considerable attention is given to the implications of dissipativeness (on a state–space representation) in terms of the existence of a set of energy-like functions. The properties of these functions are studied. To demonstrate the usefulness of the theory, Section V presents two important applications: the relation between input–output and Lyapunov stability concepts; and a structure result for passive systems which is fundamental in network synthesis. Section VI gives a survey of the known results for testing dissipativeness properties. Since this introduces computational aspects, the structure of the state space is important. In particular, only finite-dimensional systems are considered in detail. The results also provide a means whereby energy storage functions can be calculated.

II. Input–Output Theory of Dissipative Systems

2.1. *Dynamical systems*

The usual input–output description of a dynamical system is via a mapping between appropriately defined function spaces. There is some variability; e.g. one can choose to use relations or operators. Such matters are inessential to the ideas of the sequel. We will use an operator setting (**4**).

The system operator is defined on so-called signal spaces. We now give a formalism of this concept. Let τ be the set of instants of time which are of interest. Let V be an inner product space and $\mathcal{V}$ be the class of functions on τ taking their values in V. Suppose then that a real-valued inner product $\langle \cdot, \cdot \rangle$ is defined on a subset of $\mathcal{V}$ (usually derived from the inner product on V). The associated *small signal space* is given by $\mathbf{V} \triangleq \{v \in \mathcal{V} : \langle v, v\rangle < \infty\}$. We assume that $\mathcal{V}$ is a Hilbert space; an example being the space of square integrable functions $f: \mathbb{R}_+ \to \mathbb{R}^n$ denoted by $\mathbf{L}_2^n(\mathbb{R}_+)$, where $\mathbb{R}_+ = [t_0, \infty)$.

To allow for functions which are unbounded in some sense, it is convenient to extend this space of functions. We introduce the truncation operator P_T on $\mathcal{V}$ which satisfies

$$(P_T v)(t) \triangleq \begin{cases} v(t), & t \le T \\ \theta, & t > T \end{cases} \tag{1}$$

where θ denotes the zero vector in V. It is convenient to use the notation

$v_T = P_T v$. Using the truncation operator, we define the *signal space* by $\mathbf{V}_e \triangleq \{v \in \mathscr{V} : \langle v_T, v_T \rangle < \infty \forall T \in \tau\}$. Obviously, $\mathbf{V}_e$ is the space of functions whose truncations are small signals in the sense previously defined. A concept of boundedness is provided on $\mathbf{V}$ by the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. In the sequel, it is convenient if the scalar product is required to have the following properties: (P1) For all $v \in \mathbf{V}_e$, the function $T \to \|v_T\|$ is monotonically increasing and for all

$$v \in \mathbf{V}, \lim_{T \to \infty} \|v_T\| = \|v\|.$$

(P2) For all $u, v \in \mathbf{V}_e$, for all $T \in \tau$, $\langle u_T, v_T \rangle = \langle u_T, v \rangle = \langle u, v_T \rangle \triangleq \langle u, v \rangle_T$.

We can now introduce the definition of a dynamical system in input-output form. Suppose that $\mathbf{U}$ and $\mathbf{Y}$ are small signal spaces with appropriate truncation operators P_T^u and P_T^y which define the signal spaces $\mathbf{U}_e$ and $\mathbf{Y}_e$ respectively. $\mathbf{U}_e$ is called the *input signal space* and $\mathbf{Y}_e$ is called the *output signal space.*

Definition 1

A *dynamical system input–output representation* is an operator $H : \mathbf{U}_e \to \mathbf{Y}_e$.

Note that this definition does not include causality of the operator H; that is, the condition that $P_T^y H P_T^u = P_T^y H$. Such a restriction is often included in the definition for a dynamical system. However, in system theory the need to consider non-causal systems arises sufficiently often to warrant consideration of causality as a separate issue. Non-causal systems are of special importance in deriving general results on stability and instability for feedback systems (**4**).

A special class of operators which we need to consider are called memoryless. These are both causal and anticausal (**22**). A memoryless self-adjoint operator $T : \mathbf{V} \to \mathbf{V}$ is said to be positive definite (non-negative definite) if $\langle v, Tv \rangle > 0$ for all $v \in \mathbf{V}$, $v \neq 0$ ($\langle v, Tv \rangle \geq 0$ for all $v \in \mathbf{V}$). T is said to be negative definite (non-positive definite) if the inequality signs above are reversed.

2.2. *Dissipative systems*

We now define dissipativeness for the system operator H defined above. In the following definition, let $Q : \mathbf{Y}_e \to \mathbf{Y}_e$, $S : \mathbf{U}_e \to \mathbf{Y}_e$ and $R : \mathbf{U}_e \to \mathbf{U}_e$ be arbitrarily chosen memoryless bounded linear operators, with Q and R self-adjoint.

Definition 2

The dynamical system H is *dissipative* with respect to the triple (Q, S, R) iff

$$\langle y, Qy \rangle_T + 2\langle y, Su \rangle_T + \langle u, Ru \rangle_T \geq 0 \tag{2}$$

for all $T \in \tau$ and all $u \in \mathbf{U}_e$.

This definition was presented in (**18**) (with $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$ so Q, S, R could be taken as appropriately dimensioned real matrices). For present purposes, it is necessary to introduce weaker versions of this property. These have not been presented in a dissipativeness sense previously; however, some related ideas are certainly well-known in special cases. We delay for now a more complete discussion of this point.

Rather than require the inequality (2) to hold for all input signals, it is of interest to suppose that it holds only for a subset of the signal space. With this thought, we are led to attach particular significance to the set

$$\mathbf{K}(H) \triangleq \{u \in \mathbf{U} : y = Hu \in \mathbf{Y}\}.$$

In anticipation of a later discussion, this set $\mathbf{K}(H)$ is of use in formulating tests for instability—there we ask about the existence of some $u \in \mathbf{U}$ for which $y \notin \mathbf{Y}$.

Definition 3

The dynamical system H is *ultimately virtual-dissipative* with respect to the triple (Q, S, R) iff

$$\langle y, Qy \rangle + 2\langle y, Su \rangle + \langle u, Ru \rangle \geq 0 \tag{3}$$

for all $u \in \mathbf{K}(H)$. If $\mathbf{K}(H) = \mathbf{U}$, the system is said to be *ultimately dissipative*.

We note that ultimate dissipativeness applies in the special case where the system can be defined on small signal spaces as the operator $H: \mathbf{U} \rightarrow \mathbf{Y}$. For this class of system, it is obviously a weaker property than dissipativeness; roughly speaking, the system is required to be dissipative only in the limit $T \rightarrow \infty$.

In view of the established use of various names for special cases of the inequalities (2) and (3), some comments on this aspect are in order. Historically, two important special cases relate to passive and scattering operators in circuit theory and the associated concepts of positive real and bounded real matrices (**10, 22**). The relationship between this history and dissipativeness has been dealt with elsewhere (**7, 16**). Passivity corresponds to dissipativeness with respect to $(0, \frac{1}{2}I, 0)$ where I is the identity operator and we assume that $\mathbf{U} = \mathbf{Y}$. In the literature ultimate passivity has been referred to as positivity (**4, 22**). A scattering operator is dissipative with respect to $(-I, 0, I)$. Operators satisfying the ultimate dissipativeness version of this property are commonly called contractive (**22**). One unfortunate clash of naming occurs in (**22**); where, instead of the term scattering operator, the operator is said to be semidissipative. For obvious reasons, we cannot accomodate that particular nomenclature here. A slight generalization of the scattering operator definition is useful in stability theory (**4, 17**): a system is said to have finite gain if it is dissipative with respect to $(-I, 0, k^2 I)$, where k is some fixed scalar. The special forms of dissipativeness mentioned above in no way exhaust the list of useful triples (Q, S, R).

2.3. *Dissipativeness and causality*

The interrelationship between passivity, positivity and causality (and scattering operators, contractive operators and causality) is well-known (**4, 22**). For the sake of completeness, we will give the more general version of these results which relate dissipativeness to ultimate dissipativeness.

Theorem 1

Suppose that the operator H is causal and ultimately dissipative with Q non-positive definite. Then H is dissipative.

Proof: Since H is ultimately dissipative it satisfies (3) for all $u \in \mathbf{U}$. Now consider an arbitrary $u \in \mathbf{U}_e$. From (3), we can write

$$\langle HP_T^u u, QHP_T^u u\rangle + 2\langle HP_T^u u, SP_T^u u\rangle + \langle P_T^u u, RP_T^u u\rangle \geq 0.$$

Dissipativeness follows from the causality of operators Q, S, R and H and properties ($P1$), ($P2$) of the innerproduct: it is easily seen that $\langle P_T^u u, RP_T^u u\rangle = \langle u, Ru\rangle_T$ and $\langle HP_T^u u, SP_T^u u\rangle = \langle y, Su\rangle_T$. The remaining term can be treated as follows. Since Q is a non-positive definite self-adjoint operator on a Hilbert space, it can be factored according to $Q = -M^*M$ where M is a memoryless operator on $\mathbf{Y}$ and M^* denotes its adjoint operator (**23**). Then

$$\begin{aligned}\langle HP_T^u u, QHP_T^u u\rangle &= -\|MHP_T^u u\|^2 \\ &\leq -\|P_T^y MHP_T^u u\|^2 \quad \text{from } (P1) \\ &= -\|P_T^y MHu\|^2 \quad \text{from causality of } M \text{ and } H \\ &= \langle y, Qy\rangle_T.\end{aligned}$$

The inequality (2) follows.

As an immediate consequence of Theorem 1, we can say that, for a causal dynamical system $H: \mathbf{U} \to \mathbf{Y}$ and Q non-positive definite, dissipativeness and ultimate dissipativeness are equivalent. For linear operators, more explicit results can be given. It is well-known that certain types of dissipativeness imply causality of linear operators (**22**). Such results really fall outside the scope of the remainder of the paper, so we shall suppress further consideration.

2.4. *Lossless operators*

As a final consideration, we define a strong form of dissipativeness.

Definition 4

The dynamical system H is *lossless* with respect to the triple (Q, S, R) if it is dissipative and satisfies

$$\langle y, Qy\rangle + 2\langle y, Su\rangle + \langle u, Ru\rangle = 0, \tag{4}$$

for all $u \in \mathbf{K}(H)$.

There are obviously many possible variations of this definition. The significance of the concept will become clearer in later sections. Note that if $\mathbf{K}(H) = \mathbf{U}$ and the system is causal, then losslessness is equivalent to Eq. (4) holding for all $u \in \mathbf{U}$. More generally, though, we do not insist that (4) should hold for all $u \in \mathbf{U}$. To see why, consider an example where H is a single integrator; that is, $(Hu)(t) = \int_{t_0}^{t} u(\sigma)\, d\sigma$ and the small signal space is $\mathbf{L}_2(\mathbb{R}_+)$. It is easily seen that H is lossless with respect to $(0, I, 0)$ according to definition 4, which agrees with the commonly accepted intuitive idea that an integrator should be considered to be lossless. However, Eq. (4) is satisfied only when $u \in \mathbf{K}(H)$, and not otherwise.

III. State–Space Theory of Dissipative Systems

3.1. *Dynamical system representation*

In this section, we present a way in which a dynamical system can be represented in state–space form. A more complete discussion may be found in ref. (**24**). We confine our attention to continuous dynamical systems defined on the half-line $\mathbb{R}_+$. Similar descriptions apply to other choices of τ.

To maintain consistency with Section II, we denote the admissible input and output function spaces $\mathbf{U}_e$ and $\mathbf{Y}_e$ respectively; however, for purposes of the following definition the topological associations are inessential. (A usual restriction is just that $\mathbf{U}_e$ and $\mathbf{Y}_e$ be closed under concatenation). We introduce the so-called *state space* X which is just an abstract set.

Definition 5

A *dynamical system state-space representation* is defined through the sets U, $\mathbf{U}_e$, Y, $\mathbf{Y}_e$, X and the mappings ψ and r. The map $\psi:\mathbb{R}_+^2\times X\times \mathbf{U}_e\to X$ is the *state transition function*; it satisfies the axioms:

(consistency) $\psi(t_0, t_0, x_0, u)=x_0$ for all $t_0\in\mathbb{R}$, $x_0\in X$, and $u\in\mathbf{U}_e$;

(determinism): $\psi(t_1, t_0, x_0, u_1)=\psi(t_1, t_0, x_0, u_2)$ for all $(t_1, t_0)\in\mathbb{R}_+^2$, $x_0\in X$, and $u_1, u_2\in\mathbf{U}_e$ satisfying $u_1(t)=u_2(t)$ for $t_0\le t\le t_1$;

(semi-group property): $\psi(t_2, t_0, x_0, u)=\psi[t_2, t_1, \psi(t_1, t_0, x_0, u), u]$ for all $t_0\le t_1\le t_2$, $x_0\in X$, and $u\in\mathbf{U}_e$.

The map $r:X\times U\times R_+\to Y$ is the *read-out function.* It is such that the function $r[\psi(t, t_0, x_0, u), u(t), t]$ defined for $t\ge t_0$ is, for all $x_0\in X$, $t_0\in\mathbb{R}$ and $u\in\mathbf{U}_e$, the restriction to $[t_0,\infty)$ of a function $y\in\mathbf{Y}_e$. We also assume that the system is unbiased in the sense that $\psi(t, t_0, 0, 0)=0$ for all $(t, t_0)\in\mathbb{R}_+^2$ and $r(0, 0, t)=0$ for all $t\in\mathbb{R}$.

The above definition views the dynamical system through a state x which is intermediate between the input u and output y. We write the state and output at time t as $x(t)=\psi(t, t_0, x_0, u)$ and $y(t)=r[\psi(t, t_0, x_0, u), u(t), t]$ respectively. We can consider the system as a collection of trajectories in the state–space; each emanating from an initial condition and guided by the particular input. The theory of such abstract objects when X is a metric space is vast. We will refer to some of this in later considerations.

In the sequel, we shall require at times that the dynamical system be controllable and/or reachable.

Definition 6

A state $x_0\in X$ of a dynamical system in state–space form is said to be controllable at time $t_0\in\mathbb{R}$ if there exists a $t_1\ge t_0$ and a $u\in\mathbf{U}_e$ such that $\psi(t_1, t_0, x_0, u)=0$. The dynamical system is said to be *controllable* if every state $x_0\in X$ is controllable for all $t_0\in\mathbb{R}$.

Definition 7

A state $x_0\in X$ of a dynamical system in state–space form is said to be reachable at time $t_0\in\mathbb{R}$ if there exists a $t_{-1}\le t_0$ and a $u\in\mathbf{U}_e$ such that

$\psi(t_0, t_{-1}, 0, u) = x_0$. The state space of the dynamical system is said to be *reachable* if every state $x_0 \in X$ is reachable for all $t_0 \in \mathbb{R}$.

3.2. *Dissipative systems*

The concept of dissipativeness has been introduced as an input–output property for dynamical systems. In the remainder of Section III, we develop at length the implications of this property on a state–space representation. The treatment is adopted from **20** (except that in (**20**) the system is assumed to be stationary).

With reference to Section II, we now suppose that the system input–output description is based on $\mathbf{U} = \mathbf{L}_2^m(\mathbb{R}_+)$ and $\mathbf{Y} = \mathbf{L}_2^p(\mathbb{R}_+)$ where $y = Hu$ is a zero-state response. We now introduce the concept of a *supply rate*: this is the function $w : U \times Y \to \mathbb{R}$ given by

$$w(u, y) = y'Qy + 2y'Su + u'Ru, \tag{5}$$

where $Q \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$ and $R \in \mathbb{R}^{m\times m}$ are constant matrices, with Q and R symmetric. Writing $w(t) = w[u(t), y(t)]$, evaluated along the system motions, it is then obvious that inequality (2) becomes

$$\int_{t_0}^{t_1} w(t)\,\mathrm{d}t \geq 0 \tag{6a}$$

for all $t_1 \geq t_0$ and all $u \in \mathbf{L}_{2_e}^m$, whenever the initial state $x(t_0) = 0$. We usually say that the system is dissipative with respect to the supply rate $w(\cdot,\cdot)$ in this context. (Sometimes, for convenience, we will drop the phrase "with respect to supply rate $w(\cdot,\cdot)$"; it being always understood that a particular given supply rate is under consideration).

The theory to be discussed carries abstract energy connotations (as does the term dissipative itself). This derives from the interpretation of the supply rate as an input power; consequently, the inequality (6a) restricts the manner in which the system absorbs energy. Before considering this interpretation further, we make another definition.

Definition 8

A dynamical system is defined to be *cyclo-dissipative* iff

$$\int_{t_0}^{t_1} w(t)\,\mathrm{d}t \geq 0, \tag{6b}$$

for all $t_1 \geq t_0$ and all $u \in \mathbf{L}_{2_e}^m$, whenever $x(t_0) = x(t_1) = 0$.

Note that cyclo-dissipativeness does not immediately correspond to a previously defined input–output property, because of the role of the final state $x(t_1)$. However, cyclo-dissipativeness is closely related to ultimate virtual-dissipativeness. Another point to note is that the quadratic nature of $w(\cdot,\cdot)$ is inessential to Definitions 7 and 8; and also to virtually all that we present in this section. It is required only for consistency with Section II.

Theorem 2

Suppose that the system H is observable in the sense that $u \in \mathbf{K}(H)$ implies the zero-state response satisfies $\lim_{t\to\infty} x(t) = 0$. Then ultimate virtual-dissipativeness is equivalent to cyclo-dissipativeness.

Proof: Suppose that the system is ultimately virtual-dissipative. Then Eq. (3) can be written as

$$\int_{t_0}^{\infty} w(t)\, \mathrm{d}t \geq 0, \tag{7}$$

for all $u \in \mathbf{K}(H)$, whenever $x(t_0) = 0$.

Now consider any $u \in \mathbf{L}_{2_e}^{m}(\mathbb{R}_+)$ which takes the system from $x(t_0) = 0$ to $x(t_1) = 0$. Then choose $P_{t_1} u \equiv 0$. Since the system is unbiased, it follows that $P_{t_1} x \equiv 0$, $P_{t_1} y \equiv 0$ (and $u \in \mathbf{K}(H)$). Then Eq. (7) can be written as

$$\int_{t_0}^{t_1} w(t)\, \mathrm{d}t \geq 0.$$

That is, the system is cyclo-dissipative.

For the converse, suppose that the system is cyclo-dissipative. Consider any $u \in \mathbf{K}(H)$. Smoothness gives that Eq. (7) holds (considering the limit $t_1 \to \infty$ in the definition of cyclo-dissipativeness).

This theorem, with Theorem 1, gives interrelationships between the various forms of dissipativeness. As a consequence of Theorem 2, it is reasonable to use the terms virtual-dissipativeness and cyclo-dissipativeness interchangeably; the latter term has historical precedence (**14**), but really only carries intuition for state–space representations. As a general comment, the ultimate versions of dissipativeness relate the total output function to the total input function. On the other hand, dissipativeness and cyclo-dissipativeness are concerned with the time evolution of these functions as well. In a sense, the differences evanesce for causal systems; but it will be obvious in Section V how such concepts are of importance in the study of stability.

Cyclo-dissipative systems exhibit a net absorption of energy along any trajectory which starts and ends at the origin of X. A cyclo-dissipative system might, however, produce energy along some initial portion of such a trajectory; if so, it would not be dissipative. On the other hand, every dissipative system is cyclo-dissipative.

(In interpreting these comments, it should of course be recognized that the "energy" in question need have no physical significance. The term "energy" is being used here simply to mean the quantity defined by the integrals in the above definitions).

As an example, suppose that the system under consideration is an electrical network, whose elements are constant resistors, inductors and capacitors. Let $u(t)$ be the vector of port currents, and $y(t)$ the vector of corresponding port voltages. Then it may be shown [via Tellegen's theorem (**25**) for example], that the system is cyclo-dissipative with respect to supply rate $w(u, y) = u'y$, provided that all resistances are non-negative. If in addition all inductances and capacitances are non-negative, then the system is dissipative (with respect to the same supply rate).

In the above example, we can, via physical reasoning, define a stored energy for the network. For a more general abstract system, physical reasoning fails us, but we can at least define possible candidates for the name "stored energy". Consider, then, the following two functions.

Definition 9

The *required supply* $\phi_r : X \times \mathbb{R} \to \mathbb{R}_e$ is defined by

$$\phi_r(x_0, t_0) = \inf_{u \in \mathbf{U}_e, t_{-1} \leq t_0} \int_{t_{-1}}^{t_0} w(t)\,\mathrm{d}t$$

with boundary conditions $x(t_{-1}) = 0$, $x(t_0) = x_0$ where $\mathbb{R}_e = \mathbb{R} \cup \{\infty\}$ is the extended real line. The *virtual available storage* $\phi_a^* : X \times \mathbb{R} \to \mathbb{R}_e$ is defined by

$$\phi_a^*(x_0, t_0) = -\inf_{u \in \mathbf{U}_e, t_1 \geq t_0} \int_{t_0}^{t_1} w(t)\,\mathrm{d}t$$

with boundary conditions $x(t_0) = x_0$, $x(t_1) = 0$.

To simplify the following discussion, we can arbitrarily assign the value $-\infty$ to the infima if the boundary conditions can be met, but the infima fail to exist. If the boundary conditions cannot be met, we can similarly assign the "infimum" a value of $+\infty$.

The required supply is the least amount of energy required to excite a system to a given state; the virtual available storage is the maximum amount that one can extract from the system when starting from a given initial state, under the constraint that the final state must be zero. In general, there are no *a priori* bounds on these two functions—they need not be finite for any given x_0. However, we have the following (obvious) property.

Lemma 1

Regardless of dissipativeness or cyclo-dissipativeness,

(a) $\phi_r(x_0, t_0) < \infty$ for any state x_0 reachable at t_0, and

(b) $\phi_a^*(x_0, t_0) > -\infty$ for any state x_0 controllable at t_0.

Proof: Directly from the definitions of controllability and reachability.

For a cyclo-dissipative system, we have a slightly more informative result.

Lemma 2

Let the system be cyclo-dissipative. Then

$$\phi_a^*(0, t_0) = \phi_r(0, t_0) = 0$$

and

$$\phi_r(x_0, t_0) \geq \phi_a^*(x_0, t_0)$$

for any state $x_0 \in X$ and $t_0 \in \mathbb{R}$.

Proof:

If x_0 is both controllable and reachable at t_0, we have the inequality

$$\int_{t_{-1}}^{t_0} w(t)\,\mathrm{d}t + \int_{t_0}^{t_1} w(t)\,\mathrm{d}t \geq 0$$

where the trajectory is chosen to pass through the points $x(t_{-1}) = x(t_1) = 0$, and $x(t_0) = x_0$. The result then follows from the definitions of ϕ_r and ϕ_a^*. If x_0 is uncontrollable and/or unreachable, the result still holds since we have $\phi_r(x_0, t_0) = \infty \geq \phi_a^*(x_0, t_0)$ or $\phi_r(x_0, t_0) \geq \phi_a^*(x_0, t_0) = -\infty$, as appropriate.

In particular, for a controllable and reachable cyclo-dissipative system we have $-\infty < \phi_a^*(x, t) \leq \phi_r(x, t) < \infty$ for all $x \in X$, $t \in \mathbb{R}$. Notice that neither the controllability nor the reachability constraint can be dropped in obtaining this result.

Let us now define a third possible candidate for "stored energy".

Definition 10

The *available storage* $\phi_a : X \times \mathbb{R} \to \mathbb{R}_e$ is defined by

$$\phi_a(x_0, t_0) = -\inf_{u \in \mathbf{U}_e, t_1 \geq t_0} \int_{t_0}^{t_1} w(t)\,\mathrm{d}t$$

with boundary conditions $x(t_0) = x_0$, $x(t_1)$ free.

Immediately we have $\phi_a(x, t) \geq 0$, and $\phi_a(x, t) \geq \phi_a^*(x, t)$, for all x and t for which the functions are defined. (Note, too, that these inequalities do not depend on cyclo-dissipativeness). In general, cyclo-dissipativeness will not provide an upper bound for ϕ_a, nor will controllability nor reachability provide any such bound. For dissipative systems, however, we can find such a bound.

Lemma 3

Let the system be dissipative. Then

$$\phi_a(0, t_0) = \phi_r(0, t_0) = 0$$

and

$$\phi_r(x_0, t_0) \geq \phi_a(x_0, t_0) \geq 0$$

for any $x_0 \in X$ and $t_0 \in \mathbb{R}$.

Proof:

Similar to the proof of Lemma 2.

In particular, for a reachable and controllable (reachable) cyclo-dissipative (dissipative) system, we have

$$-\infty < \phi_a^*(x, t) \leq \phi_r(x, t) < \infty [0 \leq \phi_a(x, t) \leq \phi_r(x, t) < \infty]$$

for all x, t. A more detailed treatment of the implications of reachability and controllability is given in the next section. Note that controllability is of only minor interest for dissipative systems, except insofar as dissipative systems partake of the properties of cyclo-dissipatiave systems.

IV. Storage Functions

So far, we have defined three functions which might represent the stored energy of a dissipative or cyclo-dissipative system. Except in some special cases, these functions differ in value from one another. This means that, lacking any further information about the internal structure of a system, we cannot uniquely define the stored energy of the system. We can, however, describe a whole class of functions which are candidates for the name "stored energy". These functions were defined by Willems (**7, 14**) in the following way.

Definition 11

A function $\phi: X\times\mathbb{R}\to\mathbb{R}_e$ is called a *virtual storage function* if it satisfies $\phi(0, t)=0$ for all t and

$$\phi(x_0, t_0)+\int_{t_0}^{t_1} w(t)\,dt \geq \phi(x_1, t_1) \tag{8}$$

for all $t_1 \geq t_0$ and all $u \in \mathbf{U}_e$ where $x(t_0)=x_0$ and $x(t_1)=x_1$. If in addition $\phi(x, t)\geq 0$ for all x, t then ϕ is called a *storage function*. For time-invariant systems, we restrict attention to (virtual) storage functions which are independent of time t.

The constraint $\phi(0, t)=0$ is inessential, and $\phi(x, t)\geq 0$ could be replaced by $\phi(x, t)\geq\phi(0, t)$ for all x, t. That is, the addition of a non-zero $\phi(0, t)$ would leave inequality (8) unaffected. However, to simplify later discussions concerning orderings between (virtual) storage functions, it is desirable to insist from the outset that the functions have no non-zero $\phi(0, t)$ "bias" component. Actually, Definition 11 departs a little from the formulation in (**7, 14**) by requiring $\phi(x, t)\geq\phi(0, t)$ for all x, t. Briefly, the motivation for the difference is as follows: in (**7, 14**) dissipativeness is introduced as the property of a state–space representation whereby there exists a storage function. However, the present discussion has chosen to start with an input–output definition which, for instance, implies $\phi_a(x, t)\geq\phi_a(0, t)=0$ for all x, t. A further slight difference to (**7, 14**) is that to allow a treatment of nonminimal systems, we do not rule out the possibility of infinite values of the (virtual) storage functions and discuss existence in the usual sense as a separate issue.

To relate this definition to the results of the last section, we have the following results.

Lemma 4

The functions ϕ_a^* and ϕ_r are virtual storage functions for a cyclo-dissipative system. For a dissipative system, ϕ_a and ϕ_r are storage functions.

Proof: We shall show the method of proof for ϕ_r only; the other two cases are similar. First, we note $\phi_r(0, t)=0$ for all t follows from Lemma 2.

Suppose that x_0 and x_1 are reachable states. From the definition of ϕ_r, we have

$$\phi_r(x_1, t_1)\leq\int_{t_{-1}}^{t_1} w(t)\,dt$$

for any u taking $x(t_{-1})=0$ to $x(t_1)=x_1$ (provided of course that $t_{-1}\leq t_0$). In particular, let t_{-1}, t_0 and u on $[t_{-1}, t_0)$ be such that $x(t_0)=x_0$ is reached in an optimal manner. Except for the boundary conditions $x(t_0)=x_0$ and $x(t_1)=x_1$, u is still free on the time interval t_0 to t_1, and we have our result.

If x_0 and/or x_1 is unreachable, the result still holds. For example, if x_0 is reachable but x_1 is not, then x_1 cannot be reachable from x_0; in other words, $\int_{t_0}^{t_1} w(t)\,\mathrm{d}t$ cannot be finite for any transfer between x_0 and x_1. The remaining cases can be treated by similar arguments.

Theorem 3

A system is cyclo-dissipative iff there exists a virtual storage function ϕ(with $-\infty<\phi(x,t)<\infty$ for all $x\in X$ which are both controllable and reachable at $t\in\mathbb{R}$).

Proof: Suppose first that some ϕ exists such that Eq. (8) is satisfied. Then certainly $\phi(0,t)=0$ is well-defined. Setting $x_0=x_1=0$ in (8) we retrieve the definition of cyclo-dissipativeness in (6b).

Conversely, suppose that the system is cyclo-dissipative. Then from Lemma 4 ϕ_r and ϕ_a are both valid virtual storage functions. From Lemmas 1 and 2 these functions take on finite values at every point which is both controllable and reachable.

For dissipative systems, we have the following result.

Theorem 4

A system is dissipative iff there exists a storage function ϕ (with $0\leq\phi(x,t)<\infty$ for all x reachable at $t\in\mathbb{R}$).

Proof: If the system is dissipative, then Lemmas 1, 3 and 4 give that both ϕ_a and ϕ_r are storage functions satisfying the conditions of the theorem.

For the converse, suppose that some storage function ϕ exists. Setting $x_0=0$ in (8), we have

$$\int_{t_0}^{t_1} w(t)\,\mathrm{d}t\geq\phi(x_1,t_1).$$

Since $\phi(x_1,t_1)\geq 0$, it follows that the system is dissipative.

The following result is also worth noting.

Theorem 5

For a cyclo-dissipative time-invariant system,

$$\int_{t_0}^{t_1} w(t)\,\mathrm{d}t\geq 0$$

for any $x(t_0)$ which is both reachable and controllable, any $t_1\geq t_0$, and any $u\in\mathbf{U}_e$ such that $x(t_1)=x(t_0)$.

Proof: Directly from (8) and the fact that ϕ_r (and, for that matter, ϕ_a^*) is a virtual storage function which takes finite values for all controllable and reachable states.

In other words, if a time-invariant system is cyclo-dissipative—that is, if it absorbs energy for any cyclic motion passing through the origin—then it absorbs energy during any cyclic motion, provided that at least one point on the state trajectory is both reachable and controllable.

4.1. *Properties of the storage functions*

In this section, we consider some of the properties of the set of all (virtual) storage functions. Our first two results show that one can place tight upper and lower bounds on the set of all virtual storage functions, and equally tight upper and lower bounds on the set of all storage functions for a dissipative system. The latter result was given in (**7**).

Theorem 6

Let ϕ be any virtual storage function for a cyclo-dissipative system. Then

$$\phi_a^*(x, t) \leq \phi(x, t) \leq \phi_r(x, t)$$

for all $x \in X$ and $t \in \mathbb{R}$.

Proof: To establish the upper bound, let x_1 be any state reachable at t_1, and let $u \in \mathbf{U}_e$ be any control taking $x(t_0) = 0$ to $x(t_1) = x_1$. Then from Eq. (8), we have

$$\phi(x_1, t_1) \leq \int_{t_0}^{t_1} w(u, y)\, dt.$$

Since this is true for all u, we have

$$\phi(x_1, t_1) \leq \inf_{u \in \mathbf{U}_e} \int_{t_0}^{t_1} w(u, y)\, dt = \phi_r(x_1, t_1)$$

which establishes the result. (For unreachable states, there is of course nothing to prove). The lower bound is established by a similar argument, based on controls which take $x(t_0) = x_0$ to $x(t_1) = 0$, with x_0 being any controllable state.

Theorem 7

Let ϕ be any storage function for a dissipative system. Then

$$0 \leq \phi_a(x, t) \leq \phi(x, t) \leq \phi_r(x, t)$$

for all $x \in X$ and $t \in \mathbb{R}$.

Proof: The upper bound follows of course from Theorem 6. To show that $\phi \geq \phi_a$, consider any trajectory leaving $x(t_0) = x_0$. Then, from (8) we have

$$\phi(x_0, t_0) \geq \phi(x_1, t_1) - \int_{t_0}^{t_1} w(u, y)\, dt \geq \phi(x_1, t_1) + \phi_a(x_0, t_0)$$

where $x_1 = x(t_1)$. Since $\phi(x_1, t_1) \geq 0$, regardless of the value of x_1, we have our result.

Theorem 7 has an interesting corollary. Recall that $\phi_r(x, t)$ is finite precisely when x is a reachable state, but that reachability with dissipativeness is a sufficient condition for finiteness of $\phi_a(x, t)$. These conditions are not necessary and in general there appears to be no other simple conditions which guarantee that ϕ_a is well-defined. However, we may at least deduce the following result: if there exists any storage function which is finite for all $x \in X$ and $t \in \mathbb{R}$ then $\phi_a(x, t)$ is also finite for all $x \in X$ and $t \in \mathbb{R}$.

Another interesting result is that the set of all virtual storage functions, for a given system and supply rate, is a convex set.

Theorem 8

Let ϕ_1 and ϕ_2 be any two virtual storage functions for a cyclo-dissipative system. Then

$$\phi(x, t) \triangleq \alpha\phi_1(x, t) + [1 - \alpha]\phi_2(x, t)$$

is also a virtual storage function, for any scalar α such that $0 \leq \alpha \leq 1$.

Proof: If both ϕ_1 and ϕ_2 satisfy Eq. (8), then clearly ϕ as defined above will satisfy (8).

Of course, the above theorem also holds for storage functions of a dissipative system: for if ϕ_1 and ϕ_2 are both non-negative, ϕ will also be non-negative. In other words, the set of storage functions is a convex subset of the convex set of all virtual storage functions.

4.2. *Equivalences for reachable systems*

At various points in the above, the role of reachability and controllability of the state x has been carefully discussed. It is now a straightforward consequence of previous results to give some important equivalent statements of (cyclo)-dissipativeness when the system is reachable (and controllable).

Theorem 9

Assume that the system is reachable and controllable. Then the following equivalences hold:

(i) The system is cyclo-dissipative;

(ii) $\phi_r(x, t) > -\infty$ for all $x \in X$ and $t \in \mathbb{R}$ and $\phi_r(0, t) = 0$ for all $t \in \mathbb{R}$;

(iii) $\phi_a^*(x, t) < \infty$ for all $x \in X$ and $t \in \mathbb{R}$ and $\phi_a^*(0, t) = 0$ for all $t \in \mathbb{R}$.

Proof: Follows easily from Theorems 3 and 6 and Lemmas 1, 2 and 4.

Theorem 10

Assume that the system is reachable. Then the following equivalences hold:

(i) The system is dissipative;

(ii) $\phi_r(x, t) \geq 0$ for all $x \in X$ and $t \in \mathbb{R}$ and $\phi_r(0, t) = 0$ for all $t \in \mathbb{R}$;

(iii) $\phi_a(x, t) < \infty$ for all $x \in X$ and $t \in \mathbb{R}$ and $\phi_a(0, t) = 0$ for all $t \in \mathbb{R}$.

Proof: Follows easily from Theorems 4 and 7 and Lemmas 1, 3 and 4.

The results of Theorems 9 and 10 carry very reasonable interpretations. For example, a reachable system is dissipative iff we can only extract finite energy at each $x \in X$ and there exists a point (labelled the origin) from which no energy can be extracted. At this point it is convenient to note how Theorems 3, 4, 9 and 10 provide a connection between our definitions of dissipativeness and cyclo-dissipativeness, and those of Willems (**7**, **14**). In (**7**, **14**), a system is defined to be dissipative (cyclo-dissipative) if there exists a storage function (virtual storage function)—deleting our constraint $\phi(x, t) \geq \phi(0, t) = 0$, $[\phi(0, t) = 0]$ satisfying Eq. (8) and finite for all x and t. It can be seen that the differences between these alternative definitions vanish when controllability and reachability assumptions are imposed and the origin can be regarded as a point at which no energy is stored. It should be noted that some consideration of the relation between input–output and state–space formulations of dissipativeness appeared in (**21**) with results related to those given here.

4.3. *Dissipation delay*

Although cyclo-dissipativeness is essentially an input–output property of a system, the virtual storage functions are functions of the internal state. If a virtual storage function is thought of as the stored energy of the system, then different virtual storage functions can be thought of as corresponding to different internal realizations of the given input–output mapping. The meaning of the word "realization" at this point is, of course, not meant to be precise.

For any particular realization, we can define a *dissipation function* $D(x, u, t_0, t_1)$, via a "conservation of energy" equation

$$\phi[x(t_0), t_0] + \int_{t_0}^{t_1} w(t)\, dt = \phi[x(t_1), t_1] + D(x(t_0), u, t_0, t_1). \tag{9}$$

Notice that D depends only on the given input–output mapping and supply rate, and on the particular choice of the virtual storage ϕ. $D(x, u, t_0, t_1)$ represents the total energy dissipated in the time interval from t_0 to t_1 when the system is started in state x and is subjected thereafter to control u. From Eq. (8), $D(x, u, t_0, t_1) \geq 0$ for all x, u and $t_1 \geq t_0$ (assuming of course that the system is cyclo-dissipative, and that ϕ is one of its virtual storage functions).

If $[\phi, D]$ is any pair satisfying (9), we shall call $[\phi, D]$ a *realization* of the system.

Notice that the quantity

$$\int_{t_0}^{t_1} w(t)\, dt = \phi[x(t_1), t_1] - \phi[x(t_0), t_0] + D(x(t_0), u, t_0, t_1)$$

is an invariant for any given input–output mapping and supply rate, since the left side of the equation does not depend on the particular realization chosen. In particular, we have the following lemma for stationary systems.

Lemma 5

Let $[\phi_1, D_1]$ and $[\phi_2, D_2]$ be any two realizations of a time-invariant system. Then

$$D_1(x_0, u, t_0, t_1) = D_2(x_0, u, t_0, t_1)$$

for any x_0 for which both $\phi_1(x_0, t_0)$ and $\phi_2(x_0, t_0)$ are finite, any $t_1 \geq t_0$ and any u such that $x(t_1) = x(t_0) = x_0$ for some t_0 and t_1.

Proof: For any u, we have

$$\phi_1[x(t_1), t_1] - \phi_1[x(t_0), t_0] + D_1[x(t_0), u, t_0, t_1] = \phi_2[x(t_1), t_1] - \phi_2[x(t_0), t_0] + D_2[x(t_0), u, t_0, t_0]. \quad (10)$$

Setting $x(t_1) = x(t_0)$ and using time-invariance, the result follows.

What this means is that the dissipated energy, for cyclic motions only, is a function only of the input–output map, and does not depend on the particular internal realization. It might happen, though, that some realizations dissipate most of the energy in the early part of the cycle; while for others, there could be an initial period during which very little energy is dissipated. This motivates the definition below.

Definition 12

Let $[\phi_1, D_1]$ and $[\phi_2, D_2]$ be two realizations of the same system. Then $[\phi_1, D_1]$ has less *dissipation delay* than $[\phi_2, D_2]$, written $[\phi_1, D_1] \lessdot [\phi_2, D_2]$, if

$$D_1[0, u, t_0, t_1] \geq D_2[0, u, t_0, t_1]$$

for all $u \in \mathbf{U}_e$ and all $t_1 \geq t_o$.

[It is important to notice that there are three possibilities: either $[\phi_1, D_1] \lessdot [\phi_2, D_2]$ or $[\phi_2, D_2] \lessdot [\phi_1, D_1]$, or $[\phi_1, D_1]$ and $[\phi_2, D_2]$ are incomparable]. It is not hard to show that "$\lessdot$" is a partial ordering.

Theorem 11

Let $[\phi_1, D_1]$ and $[\phi_2, D_2]$ be any two realizations of a given system. Then $[\phi_1, D_1] \lessdot [\phi_2, D_2]$ iff $\phi_1(x, t) \leq \phi_2(x, t)$ for all x and t.

Proof: Obvious from Eq. (10).

Intuitively, this result is reasonable. The inequality $\phi_1 \leq \phi_2$ means, crudely speaking, that $[\phi_1, D_1]$ has a smaller storage capacity than $[\phi_2, D_2]$. The first realization will therefore tend to dissipate energy almost as soon as it is received, while the second will store the energy for a time before dissipating it.

4.4. *Example*

To consolidate the ideas of previous sections, it is appropriate to now look at an example in some detail.

It is not hard to show that the first-order system

$$\dot{x} = -x + u$$
$$y = x + \tfrac{1}{2}u \quad (11)$$

is dissipative with respect to supply rate $w(u, y) = uy$. (In other words, it is passive.) The availability storage is $\phi_a(x) = (2 - \sqrt{3})/2\, x^2$, and the required supply is $\phi_r(x) = (2 + \sqrt{3})/2\, x^2$. (In addition, it turns out that $\phi_a^* = \phi_a$: that is, every virtual storage function is a storage function.)

Let $\phi_C(x)=\frac{1}{2}Cx^2$; then ϕ_C is a quadratic storage function iff $2-\sqrt{3}\leq C\leq 2+\sqrt{3}$. The associated energy dissipation function is

$$D_C(x_0, u, 0, T)=\int_0^T [Cx^2+(1-C)xu+\tfrac{1}{2}u^2]\,\mathrm{d}t$$

$$=\int_0^T [C(x-\gamma_C u)^2+R_C u^2]\,\mathrm{d}t,$$

where $\gamma_C=(C-1)/2C$, $R_C=\frac{1}{2}-C\gamma_C^2$. Notice that $R_C\geq 0$ iff $2-\sqrt{3}\leq C\leq 2+\sqrt{3}$. For values of C outside this range, it is possible to make the "dissipation" negative.

For a given C, it is a meaningful exercise to find that u which minimizes $D_C(x_0, u, 0, T)$. For small C, the low-dissipation trajectories turn out to be those for which $\|x\|$ is decreasing. For large C, on the other hand, the low-dissipation trajectories have the property that $\|x\|$ increases with time. The extreme cases are: (a) $C=2-\sqrt{3}$, $\phi_C(x)=\phi_a(x)$. For this realization it is possible to drive the state to the origin with an arbitrarily small amount of dissipation. That is, all of the stored energy may be extracted at the terminals. However, controls driving the state from the origin to some specified nonzero final state produce a relatively large amount of dissipation. (b) $C=2+\sqrt{3}$, $\phi_C(x)=\phi_r(x)$. In this case any state is reachable from the origin with an arbitrarily small amount of dissipation. Returning the state to the origin does, however, involve a non-negligible amount of dissipation.

These are of course special cases. They do however illustrate the result that for small C the dissipation tends to be concentrated in the earlier part of a trajectory leaving the origin (and ultimately returning to the origin), and that the opposite condition holds for those realizations associated with large C.

So far, the word "realization" has been used in an abstract sense. To provide a physical example, let us suppose that the system in question is an electrical one-port network, where u is the port current and y the port voltage. The supply rate then turns out to be the actual electrical power input to the network.

A physical realization of the network is shown in Fig. 1. The stored energy is $\frac{1}{2}Cx^2$, so that in this example we have a meaningful correspondence between

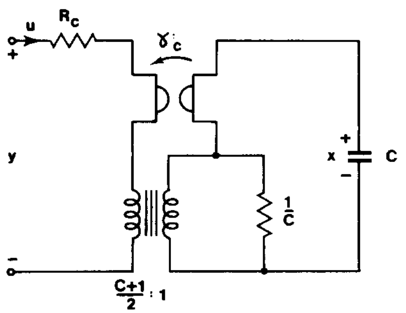

FIG. 1. A passive electrical network.

the abstract "storage function" and physical stored energy. Also, the power dissipation in the two resistors is $R_C u^2$ and $C(x-\gamma_C u)^2$, so that our earlier dissipation function does in fact represent dissipated energy. Note that for each value of C, we have a different physical realization with the state equations (11). (Setting $C=1$, it is easy to check that the network in Fig. 1 reduces to a simple reciprocal RC network).

Because our example is linear, a transfer function interpretation is possible. Let

$$y_1 = \sqrt{R_C}\,u$$
$$y_2 = \sqrt{C}(x-\gamma_C u)$$

be the two "dissipation outputs", i.e. the two resistor currents, normalized to remove the effect of the magnitudes of the resistances. Letting $Y_1(s)$, $Y_2(s)$ and $U(s)$ be the Laplace transforms of the corresponding lower-case quantities, we have

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \sqrt{R_C} \\ \sqrt{C}\dfrac{(1-\gamma_C)-\gamma_C s}{1+s} \end{bmatrix} U(s) \triangleq T(s)U(s).$$

As C varies from $2-\sqrt{3}$ to $2+\sqrt{3}$, γ_C varies monotonically from $-\frac{1}{2}(\sqrt{3}+1)$ to $+\frac{1}{2}(\sqrt{3}-1)$. The result is that the phase lag of $Y_2(s)$ with respect to $U(s)$ increases monotonically with C. Thus we have a connection between dissipation delay and phase lag. Notice also that

$$T'(-s)T(s) = \tfrac{1}{2} + \frac{1}{1-s^2}$$

independently of C. This says, in effect, that the net power dissipation rate for a periodic motion is the same for all realizations; only the phase delay is different for each different C.

The above example illustrated an excellent tie-up between our abstract concept of "realization" and an actual physical realization.

When a physical realization is chosen, this specifies an energy storage mechanism. Storage functions, on the other hand, are defined without reference to any storage mechanism. One interesting feature of the example is that $\phi_a^* = \phi_a$. One might ask whether this is true for every dissipative system. The answer is that, in general, this is not so (**26**).

4.5. *Lossless systems*

A special class of cyclo-dissipative systems are those for which no input energy is dissipated.

Definition 13

The system is defined to be *cyclo-lossless* iff

$$\int_{t_0}^{t_1} w(u, y)\,dt = 0$$

whenever $x(t_0) = x(t_1) = 0$, for all $u \in \mathbf{U}_e$ and all $t_1 \geq t_0$.

Definition 14

The system is defined to be *lossless* iff it is both cyclo-lossless and dissipative.

In view of Theorem 2, it is easy to see that for casual systems Definitions 4 and 14 agree.

The following results are obvious consequences of the above definitions.

Theorem 12

Let ϕ be any virtual storage function for a cyclo-lossless system. Then

$$\phi_a^* = \phi = \phi_r$$

and

$$\phi(x_0, t_0) + \int_{t_0}^{t_1} w(t)\,\mathrm{d}t = \phi(x_1, t_1) \tag{12}$$

for all $t_1 \geq t_0$ and all $u \in \mathbf{U}_e$, where $x(t_0) = x_0$ and $x(t_1) = x_1$.

Proof: A minor modification to the proof of Lemma 4 shows that both ϕ_a^* and ϕ_r satisfy (12). From Theorem 11 it follows that $\phi_a^* = \phi_r$. Since ϕ_a^* and ϕ_r are the minimum and maximum virtual storage functions, all virtual storage functions must be equal.

Theorem 13

Let ϕ be any storage function for a lossless system. Then

$$0 \leq \phi_a^* = \phi_a \phi = \phi_r$$

as well as (12) being satisfied.

Proof: Obvious from Theorem 11.

Notice that the only difference between lossless and cyclo-lossless systems is that in the lossless case the (unique) storage function must be non-negative.

V. Applications

Since the theory of dissipative systems evolved out of developments in stability theory and network synthesis, it is appropriate that some attention now be given to showing how basic results in those areas depend on the abstract energy concept.

For sake of brevity, we reduce generality somewhat and consider dynamical systems with non-linear finite-dimensional state–space representations of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x, u). \end{aligned} \tag{13}$$

The values of x, u and y lie in Euclidean spaces $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$. In accordance with the axioms of Definition 5 and the later assumptions that $\mathbf{U} = \mathbf{L}_2^m(\mathbb{R}_+)$ and $\mathbf{Y} = \mathbf{L}_2^p(\mathbb{R}_+)$, we assume appropriate smoothness and that $f(0, 0) = 0$ and $g(0, 0) = 0$.

Although the results to be presented apply to more general systems than (13), this setting allows a unified presentation of both applications. In keeping with the need for brevity, proofs, which are inessential to illustrating dissipative systems theory, will be omitted and references given instead.

5.1. *Preliminaries*

We proceed to collect together and sharpen some previous results to a form which facilitates the study of the specific problems of stability and synthesis. In particular, consideration needs to be given to questions of smoothness and positive definiteness of the storage functions.

In Section IV, the storage functions did not emerge with any special smoothness properties. In fact, unless forms of controllability were assumed, they could be unbounded for some finite x. Even if boundedness for all $x \in X$ is ensured, there is no guarantee that $\phi(\cdot)$ will be smooth even if functions $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are smooth. One conceptually reasonable idea for at least ensuring continuity of the storage functions is to impose a form of local controllability; this carrying the connotation of being able to reach states which are "close together" by controls which are "close together" (**15**).

Definition 15

A dynamical system is said to be locally controllable at x_0 if, for any x_1 in a suitably small open neighbourhood Ω of x_0, there exists choices of $u \in \mathbf{U}_e$ and t_1 such that the state can be driven from $x(t_0) = x_0$ to $x(t_1) = x_1$ and from $x(t_0) = x_1$ to $x(t_1) = x_0$ with the additional property that

$$\left| \int_{t_0}^{t_1} w(t)\, dt \right| \leq \rho\, (\|x_1 - x_0\|) \tag{14}$$

for some continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(0) = 0$. The dynamical system is said to be *locally controllable* if it is locally controllable at every state $x_0 \in X$.

Lemma 6

Let a dynamical system be locally controllable. Then any virtual storage function which exists for all $x \in X$ is also continuous.

Proof: Consider some arbitrary state x_0 in X and let the virtual storage function be $\phi(\cdot)$. Then for any x_1, in the neighbourhood Ω of x_0, we have from (8) that

$$\phi(x_0) + \int_{t_0}^{t_1} w(t)\, dt \geq \phi(x_1) \tag{15}$$

for the t_1 and $u \in \mathbf{U}_e$ which are specified in Definition 15. Using (14) and (15) and considering transitions in each direction between x_0 and x_1, it is easy to deduce

$$|\phi(x_1) - \phi(x_0)| \leq \rho(\|x_1 - x_0\|).$$

The arbitrariness of x_1 and continuity of $\rho(\cdot)$ give that $\phi(\cdot)$ is continuous at x_0.

For convenience in the sequel, a system which is controllable, reachable and locally controllable will be referred to as being *strongly controllable*. (For linear systems, strong controllability is equivalent to complete controllability.) From Lemma 6, it follows that strong controllability implies continuity of any virtual storage function $\phi(\cdot)$. Under these conditions the time derivative of $\phi(\cdot)$ along trajectories of (13) can be defined as follows (**5, 6**)

$$\dot{\phi}[x(t)] = \overline{\lim_{h \to 0_+}} \frac{1}{h}\{\phi[x(t+h)] - \phi[x(t)]\}.$$

We can now state a fundamental result, as a direct consequence of Theorems 3, 4 and Lemma 6.

Theorem 14

Suppose that the system (13) is strongly controllable. Then the system is cyclo-dissipative (dissipative) iff there exists a continuous function $\phi : X \to \mathbb{R}$ satisfying $\phi(0) = 0$ [$\phi(0) = 0$, $\phi(x) \geq 0$ for all x] and

$$\dot{\phi}[x(t)] \leq w[u(t), g(x(t), u(t))] \text{ for almost all } t \in \mathbb{R} \tag{16}$$

along the system trajectories.

Proof: Follows as a direct consequence of Theorems 3, 4 and Lemma 6 and integration theory. [The details are similar to a derivation in (**12**).]

This theorem is central to the application of the theory of dissipative systems. A specific connection is made between input–output and internal properties via the intuitive notion of the existence of a stored energy function. Further, the properties of this energy function are such that standard general analysis techniques— stability theory, for instance—can be immediately applied.

The use of Theorem 14 to develop stability results will evidently require the function $\phi(\cdot)$ to play the role of a Lyapunov function. Thus, we need to specify conditions which will ensure that $\phi(\cdot)$ is positive definite. As may be expected, this makes some notion of observability an issue.

Definition 16

The system (13) is said to be *zero-state detectable* if, for any trajectory such that $u(t) \equiv 0$, $y(t) \equiv 0$ implies that $x(t) \equiv 0$.

Zero-state detectability is a very weak form of observability, since it only requires that it be possible to tell if the system is in the zero state or not by observing the output. For linear systems this property is equivalent to complete observability. In addition to observability, it is convenient to pose the following property for a dissipative system.

Property A. There exists a well-defined feedback law $u^*(\cdot)$ such that $w(u^*(y), y) < 0$ for all $y \neq 0$, and $u^*(0) = 0$.

This property merely says that the input–output conditions can always be adjusted to ensure (abstract) energy flow out of the system.

The following lemma can be adapted from a result in (**16**).

Lemma 7

If the system (13) is zero-state detectable and dissipative with respect to a supply rate having property A, then all storage functions are such that $\phi(x)>0$ for all $x\neq 0$.

5.2. *The relation between input–output and Lyapunov stability properties*

The word "stability" is generally used in reference to two related, but nonetheless distinct, concepts. When used in connection with input–output representations (Section II), one talks of the boundedness of input–output mappings. Stability in a state–space context (Section III) more commonly refers to the zero-input response to a non-zero initial state. Although observations have been made about the similarity of results obtained by each method, the two approaches have largely developed independently. For dissipative systems, however, one can exhibit an especially intimate relationship between input–output stability and state–space stability; this is illustrated below.

A detailed discussion of stability definitions is beyond the scope of this paper; only a brief outline will be given. A system is called input–output stable if $y\in\mathbf{Y}$ for all $u\in\mathbf{U}$, i.e. $\mathbf{K}(H)=\mathbf{U}$. However, the stronger property of finite gain mentioned in Section II is more commonly required in stability tests: there exists a scalar $k<\infty$ such that $\|y\|_T\leq k\|u\|_T$ for all $u\in\mathbf{U}_e$ (**4**). The state–space stability concepts are those featured in standard Lyapunov stability theory of differential equations (**5**, **6**). More specifically, we will be concerned here with the property of asymptotic stability of an equilibrium state.

Recall the well-known result for linear systems: for minimal state–space representations, asymptotic and input–output stability are equivalent (**27**). The following result can be seen as a partial generalization of this to non-linear systems.

Theorem 15

The dynamical system (13) is finite gain input–output stable iff it is dissipative such that $Q<0$. Under this condition and assuming strong controllability and zero-state detectability, the null solution of the system (13) is asymptotically stable.

Proof: The proof that $Q<0$ is equivalent to input–output stability can be found in Ref. (**18**). [Of course, this does not require existence of any internal representation, let alone one of the form (13).]

The derivation of asymptotic stability depends crucially on Theorem 14. Note that, since $Q<0$, Property A holds [trivially by taking $u^*(y)\equiv 0$]. Now, invoking Theorem 14 and Lemma 7 gives the existence of a continuous function $\phi(\cdot)$ satisfying $\phi(0)=0$, $\phi(x)>0$ for all $x\neq 0$ and

$$\dot{\phi}(x)\leq g'(x,0)Qg(x,0)$$

along the zero-input trajectories of system (13). Since $Q<0$, $\dot{\phi}\leq 0$ with $\dot{\phi}=0$ only on the set $\Omega=\{x:g(x,0)=0\}$. Asymptotic stability is deduced from La Salle's Invariance Principle (**6**) using $\phi(\cdot)$ as a Lyapunov function: all trajectories in a neighbourhood of the origin approach the largest invariant set in Ω

as $t \to \infty$. But zero-state detectability implies that this invariant set consists only of the origin.

Roughly speaking, Theorem 15 tells us that any conditions leading to input–output stability plus some internal regularity constraints also imply asymptotic stability. However—in contrast to the situation for linear time-invariant systems—asymptotic stability does not in general imply finite-gain stability. Further discussion along these lines is beyond the scope of the present paper. A closely related result was derived by Willems (**13**), using a Lyapunov function not motivated by internal stored energy and without using invariance principles. The result can be seen to relate clearly the stability theory results which have evolved separately within the functional analysis approach or by using Lyapunov theory. This is particularly significant in the case where the dynamical system is actually an aggregate of a large number of interconnected subsystems. Here, a storage function $\phi(\cdot)$ can be interpreted as the summation of subsystem stored energies, and the matrix Q reflects parameters associated with subsystem dissipativeness properties and the interconnection constraints. This viewpoint leads to considerable unification of previous work as well as new results on the stability of large-scale systems (**18**).

So far, virtual-dissipativeness has not been used. However, this concept is basic to consideration of when a dynamical system is unstable. In particular, one cay say a lot about systems which are virtual-dissipative, but not dissipative. Results with an appealing parallelism to those for stability (from both the input–output and Lyapunov approach) have been derived (**14, 19**).

5.3. *Synthesis of passive systems*

A number of standard linear network synthesis techniques [see for example (**10**)] involve the decomposition of a passive network into a lossless part and a memoryless part, both of which are passive. A typical decomposition is shown in Fig. 2, where u, y_2 and u_1 are port currents and y, u_2 and y_1 are the corresponding port voltages. Anderson and Moylan (**28**) have shown that a similar decomposition may be applied to a class of nonlinear networks. It will now be shown, with the aid of Theorem 14, that the result of (**28**) continues to hold for passive systems whose state equations have the general form (13).

Assume that the system (13) is passive, that one of its storage functions $\phi(\cdot)$ is differentiable, and that the map $x \to \nabla\phi(x)$ has an inverse. That is, there exists a function $h(\cdot)$ such that $h[\nabla\phi(x)] = x$ for all x and $\nabla\phi[h(u)] = u$ for all u. Under these conditions, we claim that the original system (13) may be synthesised as a suitable interconnection of the two subsystems

$$\begin{aligned} \dot{x} &= u_1 \\ y_1 &= \nabla\phi(x) \end{aligned} \tag{17}$$

and

$$\begin{aligned} y &= g[h(u_2), u] \\ y_2 &= -f[h(u_2), u] \end{aligned} \tag{18}$$

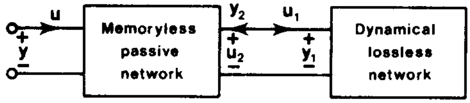

FIG. 2 Decomposition of passive network.

The interconnection is described by

$$\begin{aligned} u_1 &= -y_2, \\ u_2 &= y_1. \end{aligned} \tag{19}$$

For electrical networks, the constraints (19) represent the cascade interconnection shown in Fig. 2. For a system which is not necessarily an electrical network, Eqs. (19) still specify an interconnection which is "reasonable" in that it is both memoryless and lossless (in the sense that $u_1y_1+u_2y_2=0$). In Willems' terminology (**7**), (19) specifies a *neutral* interconnection.

It is a straightforward exercise to show that (17)–(19) are together equivalent to Eqs. (13). Further, subsystem (17) is clearly lossless and (18) is memoryless. To see that subsystem (18) is passive, notice that the result of Theorem 14 may be written as

$$\nabla'\phi(x)f(x, u)\leq u'g(x, u) \tag{20}$$

for any x. In particular, with $x = h(u_2)$, (18) and (20) together imply that

$$u'y+u_2'y_2\geq 0.$$

That is, Eqs. (18) described a passive memoryless system.

VI. Special Classes of Systems

Up to the present point in the paper, the discussion has made little reference to the structure of the dynamical system (aside from minor constraints on the spaces $\mathbf{U}_e$, $\mathbf{Y}_e$, and X, the importance of causality and the inessential restriction to finite-dimensional systems for the applications). This demonstrates that the basic ideas are independent of such structural considerations as dimensionality, time-invariance and linearity. However, testing whether a given system is dissipative in applications will obviously depend on a knowledge of system structure. Thus, we are led to a consideration of the computational—as distinct from the previous conceptual—aspects of dissipative systems.

Historically the conceptual side has been immersed in the structure required for computational feasibility. The first result to consider is the very important Kalman–Yacubovich Lemma (**8, 9**), which characterized a rational positive real transfer function (and so a single-input–single-output passive finite-dimensional linear system) in terms of the solution of a set of algebraic equations involving the matrices of a state–space realization. It is easy to interpret this solution in the stored energy function context. Subsequently many generalizations were made to more general classes of systems; the highlights being Anderson's extension to multivariable systems (**10, 29**) and Moylan's extension of the latter result to a broad class of non-linear systems (**15**). In each case, the applications were restricted to systems for which the result was derived; but, in hindsight and via the theory of dissipative systems, we see that conceptually these applications can be studied at a very general level via results such as Theorem 14.

In this section, a brief survey is given of results on characterizing dissipativeness for finite-dimensional systems. The exposition is by no means comprehensive; it being intended to illustrate the spirit of such results. For instance, results can be given for linear infinite-dimensional systems (**11**, **22**).

6.1. *Linear finite-dimensional systems*

Suppose that the dynamical system has a state–space representation of the form

$$\begin{aligned} \dot{x}(t) &= F(t)x(t) + G(t)u(t) \\ y(t) &= H'(t)x(t) + J(t)u(t) \end{aligned} \tag{21}$$

and assume that (F, G) is completely controllable. The result to be presented is adapted from Ref. (**30**). Smoothness restrictions are needed on functions $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, and $J(\cdot)$, but a precise discussion of this issue will not be presented; see Ref. (**30**) for details.

Theorem 16

A necessary and sufficient condition for the system (21) to be cyclo-dissipative (dissipative) with respect to supply rate (5) is that there exists matrices $P(\cdot)$, $L(\cdot)$ and $W(\cdot)$ with $P(\cdot)$ symmetric (non-negative definite symmetric) satisfying

$$\begin{aligned} \dot{P}(t) + P(t)F(t) + F'(t)P(t) &= H(t)QH'(t) - L(t)L'(t) \\ P(t)G(t) &= H(t)(QJ(t) + S) - L(t)W(t) \\ R + S'J(t) + J'(t)S + J'(t)QJ(t) &= W'(t)W(t). \end{aligned} \tag{22}$$

The quadratic virtual storage functions, as defined in Definition 11, are given by $\phi(x, t) = x'P(t)x$. For time-invariant systems, the storage functions do not depend explicitly on time. Of course, it could happen that the system (21) is time-varying, but the energy storage mechanism does not reflect this. (In such a case, we expect the energy dissipation to have a time-varying character.) Anyway, for time-invariant systems, the transfer function matrix is a convenient alternative description to (21). Dissipativeness can be expressed in terms of a frequency domain inequality constraint on this matrix (**7**, **20**); this being a generalization of the well-known association of positive real (generalized positive real) matrices with passive (cyclo-passive) systems (**10**).

For time-varying systems, the role of the transfer function matrix is taken by the impulse response matrix

$$Z(t, \tau) = J(t)\delta(t-\tau) + H'(t)\Phi(t, \tau)G(\tau)1(t-\tau). \tag{23}$$

In Eq. (23), $\delta(\cdot)$ and $1(\cdot)$ are respectively, the unit impulse and the unit step function, and $\Phi(\cdot,\cdot)$ is the transition matrix of $F(\cdot)$. Dissipativeness can be interpreted as an integral inequality constraining $Z(\cdot,\cdot)$. Thus dissipativeness (and cyclo-dissipativeness) can be fully characterized in both an input–output sense and a state-space sense. The solution of (22) can be achieved via the well-known Riccati differential equation (**30**).

6.2. *Nonlinear finite-dimensional systems*

As mentioned previously, Moylan (**15**) gave a characterization of passivity for a broad class of nonlinear systems. The systems referred to have a state–space representation of the form

$$\begin{aligned} x &= f(x)+G(x)u, \\ y &= h(x)+J(x)u. \end{aligned} \tag{24}$$

A generalization of the result to arbitrary dissipativeness properties has been studied in (**16, 20**); and this will now be discussed. We note that the system (24) is time-invariant and linear in the control. It has been suggested that this latter constraint can be applied without substantial loss of generality with respect to the form (13) (**31**). Also, by physical reasoning, it can be argued that time-invariance is not overly restrictive; that is, most physical circuits and systems, if adequately modelled, are time-invariant. Thus, we conclude that Eqs. (24) represents a very large class of non-linear systems. Once again, we will not discuss at length *a priori* smoothness assumptions; it suffices to say that (strong) controllability of the system and differentiability of a storage function are needed. The following result can be derived using Theorem 14 (**16, 20**).

Theorem 17

A necessary and sufficient condition for the system (24) to be cyclo-dissipative (dissipative) with respect to supply rate (6) is that there exist functions $\phi: X\to\mathbb{R}$, $l: X\to\mathbb{R}^q$ and $W: X\to\mathbb{R}^{q\times m}$, for some integer q, satisfying $\phi(0)=0$ ($\phi(0)=0$, $\phi(x)\geq 0$ for all x) and

$$\begin{aligned} \nabla'\phi(x)f(x) &= h'(x)Qh(x)-l'(x)l(x) \\ \tfrac{1}{2}G'(x)\nabla\phi(x) &= \hat{S}'(x)h(x)-W'(x)l(x) \\ \hat{R}(x) &= W'(x)W(x) \end{aligned} \tag{25}$$

for all x, where

$$\hat{S}(x)=QJ(x)+S$$

and

$$\hat{R}(x)=R+J'(x)S+S'J(x)+J'(x)QJ(x).$$

The (differentiable) storage functions are provided by the functions $\phi(\cdot)$. In these special cases, we can make (9) more explicit; as an illustration, corresponding to (25)

$$D(x(t_0),u,t_0,t_1)=\int_{t_0}^{t_1}(l(x)+W(x)u)'(l(x)+W(x)u)\,\mathrm{d}t. \tag{26}$$

At the present time, Theorem 17 provides the only general procedure for testing the dissipativeness of non-linear systems. By comparison with the linear case, this indicates that further extensions should be made in this theory. For instance, incorporating the Volterra representation results for systems of the form (24) (**2**), could lead to useful input–output characterizations of dissipativeness. A further line of consideration is to provide more explicit characterizations of the storage functions $\phi(\cdot)$ for classes of non-linear systems intermediate between linear and nonlinear of the form (24). As an illustration, the following example suggests that the interesting virtual storage functions for a class of bilinear systems (**2**) are quadratic.

Example

Consider the single-input–single-output system in which the input enters bilinearly and the output is quadratic

$$\begin{aligned} \dot{x} &= Ax + Bxu \\ y &= x'Cx. \end{aligned} \tag{27}$$

It is easy to see from Theorem 17 that system (27) is cyclo-passive (passive) if there exists a matrix P with P symmetric (non-negative definite symmetric) satisfying

$$PA + A'P \leq 0$$

$$PB + B'P = C.$$

The quadratic storage functions are given by $\phi(x) = x'Px$. A tighter study of systems like (27), along the lines known for linear systems (**10**), requires some advances in the theory of non-linear optimal control (**32**).

VII. Conclusions

The properties of dissipativeness and cyclo-dissipativeness (or virtual-dissipativeness) have been presented as input–output properties of a general dynamical system. We have seen in Theorems 3 and 4 (and their differential version Theorem 14) that, given a state–space representation, one can deduce the existence of a convex set of functions which have the appealing interpretation of stored energy. These results are seen to generalize the essential part of the well-known Kalman–Yakubovich (or Positive Real) Lemma (**8, 9, 29, 15**) to general dynamical systems. It is argued that this relation between input–output and state–space representations constitutes a fundamental basis for the study of many non-linear systems theory problems.

The applicability of the theory of dissipative systems should be considered at two levels: firstly, at the level of general dynamical systems it is a useful tool for the derivation of results on qualitative behaviour such as stability; secondly, with specialization to a particular class of systems, results which enable computation of the storage functions can be used for quantitative analysis (or synthesis). In the first category, results such as Theorem 14 play a central role; whereas Theorems 16 and 17 illustrate tools for the second category. In discussing applications, attention has been given to stability theory and non-linear system synthesis. However, applications have also been reported in optimal control and filtering theory [see (**7**, **10**) for a summary of applications to linear systems and (**15**) for some generalizations to non-linear systems]. This list of problems would not seem to exhaust the potential applicability of the theory. In particular, one of the authors is currently considering the use of dissipativeness ideas in automata theory.

Further extensions of the results of this paper could be made by considering new refinements of dissipativeness properties. For example, the input–output stability literature makes use of so-called incremental passivity properties (**14**); or, in other words, some notion of local passivity. The implications of this property on a state–space representation are of interest in the study of non-linear systems. In recent work in the field of non-linear circuits, Chua and Green (**33**) and Matsumoto (**34**) reveal clearly the importance of a property called eventual passivity; this property is intermediate between passivity and ultimate passivity and is useful for deriving boundedness results. We should also note here that the work of Willems (**7**) has recently inspired a careful study of the notion of passivity for non-linear circuits (**35**). The authors of (**35**) conclude that the input–output definition of passivity used in control theory, and adopted as a starting point in this paper, leads to results of insufficient generality for circuits. Roughly speaking, problems arise because it is not always reasonable to attach preferred status to the origin (or any other single point) as a basis for the definition of passivity; consider commonly used circuits which have multiple equilibria (**36**), or the possibility that a circuit may have no finite equilibria. To cover these situations it is proposed in (**35**) that the condition $\phi_a(x)<\infty$ for all x is a more generally reasonable definition of passivity. This work appears to provide some interesting insights into the application of dissipative systems theory.

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