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# Stability Tests for Multimachine Power Systems

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SUMMARY A method is described for generating Lyapunov functions for multi machine power systems. It is based on recent work on stability tests for interconnected dissipative control systems.

## 1 INTRODUCTION

An important aspect of the design of an electrical power system is its stability properties. It is desirable that the system be able to retain synchronism after disturbances that are likely to occur. Such behaviour becomes more complex with systems consisting of a large number of machines. Powerful analytical tools are required to aid in the understanding of such phenomena; and also in the development of design techniques. One such technique which has met with success is Lyapunov's direct method (Hahn, 1967). The present paper describes a simple approach for generating Lyapunov functions for multimachine power systems. It is based on recent work by the authors (Moylan and Hill, 1976) on stability tests for interconnected dissipative control systems.

The use of energy functions has long been recognized as a useful aid in the stability analysis of power systems. The well-known equal-area criterion for a single machine is an early example of this (Kimbark, 1956). Since Lyapunov functions are a generalization of the idea of energy, it is not surprising that much attention has been given to finding such functions. Initiated by Aylett (1958), the Lyapunov technique soon became the dominant mathematical tool for handling the multimachine problem. Interesting surveys of all this work are provided by Willems (1971) and Ribbens-Pavella (1971). Possibly the most sophisticated approach is based on formulating the problem as one of stabilizing a single loop nonlinear feedback system. Then a generalization of the Popov criterion due to Moore and Anderson (1968) suggests a suitable Lyapunov function (Willems, 1971). With regard to the multimachine stability problem, the approaches used so far have not had much success in incorporating machine models of higher than second order. This would appear to leave much room for improvement. The chief difficulty seems to arise from treating the power system as a single system of high dimension. Also techniques of insufficient flexibility for generating Lyapunov functions have often been employed.

In contrast to the previous work, this paper treats a power system as it naturally occurs: an interconnection of a number of subsystems - namely the machines. Each machine is shown to possess a property called dissipativeness. As defined in (Hill and Moylan, 1975), this is an input-output property which generalizes the well-known concept of passivity. The key idea from (Hill and Moylan, 1975) is that one can associate a function with a dissipative system; this having properties like stored energy. (These abstract energy functions turn out to be a considerable generalization of the physical energy functions used in early work on power systems). When a number of dissipative systems are interconnected, the sum of the individual stored energy functions is a possible Lyapunov function establishing stability of the overall system. It is shown in (Moylan and Hill, 1976) that this method leads to stability tests which are frequently less conservative than those hitherto available.

The structure of the paper is as follows. Section 2 contains a brief review of the theory of dissipative systems. Some of the results in (Moylan and Hill, 1976) need to be extended slightly for the present purposes; this is done in Section 3. The usual power system model is formulated as an interconnection of subsystems - each subsystem representing a machine-in Section 4. Sections 5 and 6 look at the stability of the power system with respect to small and large disturbances - that is, steady-state and transient stability. The results of Section 3 suggest the Lyapunov function that is used. Checking steady-state stability simply requires testing an NxN matrix for positive semidefiniteness, where $N$ is the number of machines. Checking transient stability is more tedious, but still computationally feasible. The Lyapunov function is essentially the same as those derived by Willems (1971) and Mansour (1972). Consequently, the results obtained represent a reinterpretation of known ideas in terms of a general approach. However, in Section 7, it is proposed that a virtue of the present approach is to facilitate the incorporation of more sophisticated machine models. Such effects as amortisseur damping, governor action and voltage regulator action may be accounted for by considering different forms of dissipativeness. From this point of view, the paper is certainly incomplete; it only represents a first step by the authors in work on developing new stability criteria for more accurate power system models.

## 2 DISSIPATIVE SYSTEMS

Consider a dynamical system whose input is an m-vector $u$ and whose output is a p-vector $y$. The precise internal description of the system is of only marginal relevance to the present discussion, but for the sake of concreteness it will be assumed that the system has state equations

$$\dot{x} = f(x,u) \quad y = h(x,u) \qquad (1)$$

where $x$ is the state vector. In addition, suppose that (1) satisfies some regularity conditions roughly summarized as controllability, weak observability, and smoothness. Associated with (1), we define a scalar function $w(\cdot,\cdot)$ of the input and output called a supply rate. This function represents an abstract power input. For the purposes of this paper it is not essential to be more specific on the mathematical constraints imposed on (1); (Hill and Moylan, 1975) can be consulted for the details.

The system (1) is said to be dissipative with respect to supply rate $w(\cdot,\cdot)$ if the inequality

$$\int_{t_0}^{t_1} w[u(t),y(t)]dt \geq 0 \qquad (2)$$

holds whenever $x(t_0) = 0$, for all $t_1 \geq t_0$ and for all $u$. In an abstract sense, inequality (2) expresses the fact that the system (1) can never supply more energy than it receives. It is important, however, to note that the words "power" and "energy" as used here do not necessarily refer to physical power and energy. Indeed, we shall be looking at systems (synchronous generators) which physically act as sources of energy; but with an appropriate choice of $w(\cdot,\cdot)$, they can be called dissipative in the sense of the above definition.

A key property of dissipative systems is the following.

Lemma 1. If system (1) is dissipative in the sense (2), there exists a function $\phi(x)$ such that $\phi(0) = 0$, $\phi(x) > 0$ for all $x \neq 0$, and

$$\phi[x(t_0)] + \int_{t_0}^{t_1} w(u,y)dt \geq \phi[x(t_1)]$$

for any $t_1 \geq t_0$, any $x(t_0)$, and any $u$.

A proof of Lemma 1 may be found in (Hill and Moylan, 1975). The function $\phi(\cdot)$ is called a storage function; in an abstract sense it represents the stored energy of the system.

Under appropriate smoothness assumptions (which are almost always met in practice), the inequality of Lemma 1 can be differentiated to give

$$\frac{d\phi[x(t)]}{dt} \leq w(u,y) \qquad (3)$$

This suggests that $\phi(x)$ might be used as a Lyapunov function (Hahn, 1967).

## 3 GENERAL STABILITY CRITERIA

In this section, we give two general stability results for large-scale systems. They are slight extensions of some results in (Moylan and Hill, 1976).

Suppose that a large system is formed by interconnecting $N$ subsystems, each of which is dissipative with respect to some supply rate and proper in the sense that $y$ is not an explicit function of $u$; specifically, let the $i^{th}$ subsystem, with input $u_i$ and output $y_i$, be dissipative with respect to

$$w_i(u_i,y_i,\dot{y}_i) = y_i'Q_iy_i + 2y_i'S_iu_i + u_i'R_iu_i + \dot{y}_i'T_iu_i \qquad (4)$$

where $Q_i$, $S_i$, $R_i$ and $T_i$ are constant matrices, $Q_i$ and $R_i$ being symmetric. (By redefining the output equation of each subsystem to include $\dot{y}_i$, this supply rate becomes of the form described in Section 2 because each subsystem is proper). Define the block diagonal matrices $Q = \text{diag}(Q_1,Q_2\ldots,Q_N)$, $S = \text{diag}(S_1,S_2,\ldots,S_N)$, $R = \text{diag}(R_1,R_2,\ldots,R_N)$ and $T = \text{diag}(T_1,T_2,\ldots,T_N)$. Also let the subsystem inputs, states and outputs be collected into column vectors $u = \text{col}(u_1,u_2,\ldots,u_N)$, $x = \text{col}(x_1,x_2,\ldots,x_N)$ and $y = \text{col}(y_1,y_2,\ldots,y_N)$.

If the overall system is formed as a linear interconnection of the subsystems, a very simple stability criterion can be derived.

Theorem 1. Let the subsystems be interconnected via the constraint $u = -Hy$. Then the overall system is stable in the sense of Lyapunov if the matrices

$$\hat{Q} \triangleq SH + H'S' - Q - H'RH \qquad (5)$$

and $TH$ are symmetric and nonnegative definite.

Proof. Let $\phi_i(x_i)$ be the storage function for subsystem $i$ (see Lemma 1). Define

$$V(x) = \sum_{i=1}^{N} (\phi_i(x_i) + \tfrac{1}{2} y'THy) \qquad (6)$$

Then from inequality (3) and after a little manipulation it turns out that

$$\dot{V}(x) \leq -y'\hat{Q}y$$

This shows that $V(x)$ is a Lyapunov function for the overall system.

For nonlinear interconnections, a similar result holds.

Theorem 2. Let the subsystems be interconnected via the constraint $u = -\psi(y)$. Then the overall system is stable in the sense of Lyapunov if $\psi(\cdot)$ satisfies

$$\sigma'Q\sigma - 2\sigma'S\psi(\sigma) + \psi'(\sigma)R\psi(\sigma) \leq 0 \qquad (7)$$

for all $\sigma$, $\psi(0) = 0$, and $T\psi(\cdot)$ is the gradient of a nonnegative function.

The proof is similar to the linear case, using

$$V(x) = \sum_{i=1}^{N} \phi_i + \int_0^y [T\psi(\sigma)]'d\sigma \qquad (8)$$

as a Lyapunov function. It should be noted that the inequality of Theorem 2 is at most quadratic in $\psi$ (linear if $R=0$), and is therefore quite easy to check.

To apply these results, one needs to be able to check dissipativeness (i.e. calculate the $Q_i$, $S_i$, $R_i$ and $T_i$) of the subsystems. For linear subsystems, the easiest method is to draw a Nyquist plot for the subsystem, and to check whether this plot avoids a certain circle in the s-plane (Zames, 1966). For many nonlinear subsystems, a method of (Hill and Moylan, 1975) can be used; this method incidentally also allows calculation of the $\phi_i(x_i)$. The situation when $T_i \neq 0$ is often more complicated, but as shown later in this paper there are at least some situations where the checking is straightforward.

## 4 MATHEMATICAL MODEL FOR THE POWER SYSTEM

We consider the situation of N synchronous round rotor machines interconnected by transmission lines. The usual assumptions used in post-fault stability studies are as follows (Ribbens-Pavella, 1971):

(1) A synchronous machine is represented by a constant voltage E behind its transient reactance. In other words, the effects of flux decay and voltage regulation are not included.

(2) Damping power for each machine is proportional to its slip velocity. (This accounts for asynchronous damping only to a rough approximation).

(3) Mechanical power input $P_m$ to each machine during the disturbance is constant; that is, the speed governors have relatively large time constants.

(4) The transfer conductances of the transmission line network are negligible.

Then the motion of the $i^{th}$ machine is given by

$$M_i\ddot{\delta}_i + a_i\dot{\delta}_i + G_iE_i^2 + \sum_{\substack{j=1\\j\neq i}}^{N} E_iE_jY_{ij}\sin(\delta_i-\delta_j)-P_{m_i}=0 \qquad (9)$$

where $M_i$, $a_i$, $G_i$ and $\delta_i$ are the inertia constant, damping constant, short-circuit conductance, and load angle of the machine respectively. $Y_{ij}$ is the modulus of the transfer admittance between the $i^{th}$ and $j^{th}$ machines.

For stability analysis, it is convenient to have a state space model for the system. This has been the subject of much discussion in the literature; especially regarding the dimension of the state space and the nature of the equilibrium points (Ribbens-Pavella, 1971). It is convenient to assume that at least one of the $a_i$ is nonzero. Then, after an appropriate redefinition of the reference speed, Willems (1974) has shown that the equilibrium of interest is given by

$$\delta_i - \delta_j = c_{ij}, \qquad \dot{\delta}_i = 0 \qquad (10)$$

where the $c_{ij}$ are constants. The natural choice of state variables is

$$x_{1_i} = \delta_i - \delta_{i_0}, \qquad x_{2_i} = \dot{\delta}_i \qquad (11)$$

where $\delta_{i_0}$ is the equilibrium value of $\delta_i$. Then the $i^{th}$ machine has the state model

$$\dot{x}_i = F_ix_i + g_iu_i \qquad (12)$$
$$y_i = h_i'x_i$$

where $F_i = \begin{bmatrix}0 & 1\\ 0 & \frac{-a_i}{M_i}\end{bmatrix}$, $g_i = \begin{bmatrix}0\\ \frac{1}{M_i}\end{bmatrix}$, and $h_i = \begin{bmatrix}1\\0\end{bmatrix}$.

For convenience, we define $b_{ij} = E_iE_jY_{ij}$.
The nonlinear interconnections are then described by $u_i = -\psi_i(y_i)$ where

$$\psi_i(\sigma) = \sum_{\substack{j=1\\j\neq i}}^{N} b_{ij}[\sin(\sigma_i-\sigma_j+c_{ij}) - \sin c_{ij}] \qquad (13)$$

Equations (12), (13) describe a system in the form discussed in Section 3. The dimension of the state space is 2N. It is now well-known that asymptotic stability has to be considered on spaces of lower dimension (Willems, 1974); to avoid such technical considerations here, we will derive a Lyapunov function for stability only in $R^{2N}$. Such a function can be simply adjusted for asymptotic stability considerations.

## 5 STEADY-STATE STABILITY

We will be concerned with the stability of the multimachine power system in a post-fault phase. In the power systems literature (Kimbark 1956), there are two notions of stability which are usually considered: steady-state and transient stability. These deal with the ability of the synchronous machines to maintain synchronism after small and large disturbances respectively. Mathematically, these correspond to local stability and stability in the sense of Lyapunov (Hahn, 1967).

In the present section, a study of steady-state stability will be undertaken by considering a linearized version of the power system model. A simple stability test is achieved by using Theorem 1.

An investigation of the Nyquist plot for (12) shows that

$$\mathrm{Re}(1+q_ij\omega)G_i(j\omega) \geq 0$$

where $G_i(s) = h_i'(sI-F_i)^{-1}g_i$ and $q_i \geq \frac{M_i}{a_i}$. This suggests a supply rate for the $i^{th}$ machine as

$$w_i(u_i,y_i,\dot{y}_i) = y_iu_i + q_i\dot{y}_iu_i \qquad (14)$$

Now dissipativeness of (12) with respect to supply rate (14) is equivalent to the passivity of this same system with the output equation altered to $y_i = x_{1_i} + q_ix_{2_i}$. Using techniques described in (Hill and Moylan, 1975) it is readily checked that an energy storage function for the $i^{th}$ machine is $\phi_i(x_i) = \frac{1}{2}x_i'P_ix_i$ where

$$P_i = \begin{bmatrix}a_i & M_i\\ M_i & q_iM_i\end{bmatrix} \qquad (15)$$

Now the linearized version of (13) is

$$\psi_i(\sigma) = k_i\sigma_i - \sum_{\substack{j=1\\j\neq i}}^{N} b_{ij}\cos c_{ij}\cdot\sigma_j$$

where $k_i = \sum_{\substack{j=1\\j\neq i}}^{N} b_{ij}\cos c_{ij}$.

Hence, in terms of the notation of Section 3, we can define the interconnection matrix H by

$$h_{ij} = \begin{cases}-b_{ij}\cos c_{ij} & , i\neq j\\ k_i & , i=j\end{cases} \qquad (16)$$

From Theorem 1, a stability test follows in terms of the positive semi-definiteness of H. The Lyapunov function which establishes stability is obtained from (6), but will not be given explicitly here; it is a special case of that used for transient stability in the next section.

The result of the stability test is an estimate of the set of equilibrium load angles about which the system operates in a stable manner. This defines the so-called steady-state stability limits of the machines (Kimbark, 1956).

## 6 TRANSIENT STABILITY

Having established that an equilibrium point is stable with respect to small disturbances, we consider the extent of the transient stability region in the state space. A post-fault initial condition in this region ensures that the swing curves of the generators do not diverge indefinitely with respect to each other and that a synchronous operating point is achieved (not necessarily at the nominal frequency). One useful technique is to use a Lyapunov function; an estimate of the transient stability region being given by where the conditions $V(x) \geq 0$, $\dot{V}(x) \leq 0$ apply.

For the system defined by (12), (13), Theorem 2 suggests a Lyapunov function of the form

$$V(x) = \tfrac{1}{2}\sum_{i=1}^{N} x_i' P_i x_i + \int_0^y [T\psi(\sigma)]' d\sigma \qquad (17)$$

where appropriate $P_i$ are given by (15).

The stability constraint on $\psi(\cdot)$ given by (7) is

$$\psi'(\sigma)\sigma \geq 0 \qquad (18)$$

for all $\sigma$. For simplicity, we let

$$q_i = q \geq \max_i \frac{M_i}{a_i} \qquad (19)$$

Then (18) ensures nonnegativity of the integral in (17) and it is easily checked that $\psi(\cdot)$ is the gradient of a real valued function (Apostel, 1957).

The line integral in (17) is given by

$$\int_0^y \psi'(\sigma)d\sigma = \sum_{i=1}^{N} \int_0^{x_{1_i}} \psi_i(0,\ldots,\sigma_i,\ldots,x_{1_n})d\sigma_i$$

$$= \sum_{i=1}^{N}\sum_{j=i+1}^{N} b_{ij}[\cos c_{ij} - \cos(x_{1_i}-x_{1_j}+c_{ij})$$

$$- \sin c_{ij}.(x_{1_i}-x_{1_j})] \qquad (20)$$

where the second line is obtained using (13) from a straightforward, but somewhat tedious, calculation.

Substituting (15) and (20) into (17) gives

$$V(x) = \sum_{i=1}^{N} \{\tfrac{1}{2}a_i x_{1_i}^2 + M_i x_{1_i} x_{2_i} + \tfrac{1}{2}M_i q x_{2_i}^2$$

$$+ q\sum_{j=i+1}^{N} b_{ij}[\cos c_{ij} - \cos(x_{1_i}-x_{1_j}+c_{ij})$$

$$- \sin c_{ij}.(x_{1_i}-x_{1_j})]\} \qquad (21)$$

This function is essentially the same as those obtained by Willems (1971) and Mansour (1972). However, it should be realized that it was derived via a very general technique. A different choice of supply rate to (14) for each machine would give a different Lyapunov function. For example, it is easy to check that the system (12) is also dissipative in the sense of exterior conicity (Zames, 1966). With reference to the above derivation, some flexibility was lost in the choice of $q_i$ to satisfy (19). This can be relaxed, but care is needed to ensure that the line integral in (17) is well-defined. The details of these investigations will not be presented here.

The use of Lyapunov functions to calculate regions of stability has been well studied (Willems and Willems, 1970). This gives estimates of transient stability limits (Kimbark, 1956) for the machines. The design of circuitry to clear faults must ensure that the power system does not operate outside these stability limits.

## 7 EXTENSIONS

The results of Sections 5 and 6 serve to illustrate a general technique for deriving Lyapunov functions for multimachine power systems. However, like previous work, they rely on a rather crude mathematical model - note the many effects listed in Section 4 that have been ignored. In this section, a brief discussion is given of how these higher order effects can be included.

For the purposes of a simple illustration, it is convenient to consider the effect of governor action. The governor dynamics for each machine can be modelled by a single time constant between the slip velocity and the mechanical power input. This suggests the model

$$\tau_i \frac{dP_{m_i}}{dt} + P_{m_i} = P_{m_i}^{o} - \eta_i \dot{\delta}_i \qquad (22)$$

Introducing the extra state variable $x_{3_i} = P_{m_i} - P_{m_i}^{o}$ to those given by (11), the model for each machine is now defined by

$$F_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{a_i}{M_i} & \frac{1}{M_i} \\ 0 & -\frac{\eta_i}{\tau_i} & \frac{-1}{\tau_i} \end{bmatrix}, \quad g_i = \begin{bmatrix} 0 \\ -\frac{1}{M_i} \\ 0 \end{bmatrix}, \text{ and } h_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is obviously a third order generalization of (12). The nonlinear interconnection equation (13) remains true. With this model it is easy to check that the supply rate (14) can still be used. Hence, by ignoring governor action, at worst we obtain an overly conservative stability criterion. However, a better result can be obtained by choosing supply rates which match the higher order system in a "tighter fashion". The advantages of the approach should now be clear: the large-scale system formulation allows concentration on the properties of each machine and the concept of dissipativeness offers a flexible way of describing these properties.

Further work on the previously mentioned and other possible extensions remains to be done. Allowing for voltage regulation requires use of Park's equations (Kimbark, 1956) to develop a model and introduces a further state feedback loop. The interconnection equations become more complex on allowing rigorously for the effects of asynchronous torques and transfer conductances.

The present discussion is meant only to indicate the direction of research on a new approach to solving these problems.

## 8 CONCLUSIONS

The results of this paper are believed to offer a useful framework within which the stability theory of multimachine power systems can be considered. The essential features of this approach are the use of a large-scale system model and the use of the concept of dissipativeness to describe the properties of each subsystem. The large-scale system approach has two important advantages over the aggregate model approach adopted in previous work. Firstly, the power system arises as an interconnection of dissipative systems - that is, the machines. By preserving this structure in the model, important aspects of machine behaviour are not obscured. Secondly, it is more computationally attractive to test the dissipativeness of a number of similar low order subsystems than do the corresponding calculation for an aggregate model.

No significantly new stability criteria have been presented in this paper; rather the novelty lies in the technique which is used. However, it is believed that this technique will lead to new results when more sophisticated models are used for the machines in the power system.

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