where $H_0 > 0$ is the maximum over all $P \in \Omega$, $\epsilon \in [0, \bar{\epsilon}]$, and over $i = 1, 2, \cdots$ of the right-hand side of (12). Since $h_k(\epsilon_{k-1}) = g_{k-1}(P_{k-1}) \leqslant \sigma$ and $h_k(\epsilon_k) = 0$,

$$0 \leqslant H_0(\epsilon_k - \epsilon_{k-1}) + h_k(\epsilon_{k-1}) \leqslant H_0(\epsilon_k - \epsilon_{k-1}) + \sigma \tag{14}$$

or

$$\epsilon_k - \epsilon_{k-1} \geqslant -\frac{\sigma}{H_0} > 0 \tag{15}$$

which contradicts the requirement that ϵ_k be bounded. It is not difficult to show that there exist no $\tilde{\epsilon} > \epsilon_{\text{opt}}$ and $\tilde{P}$ that ensure stability for all allowable parameter variations. Q.E.D.

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The Stability of Nonlinear Dissipative Systems

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***Abstract*—This short paper presents a technique for generating Lyapunov functions for a broad class of nonlinear systems represented by state equations. The system, for which a Lyapunov function is required, is assumed to have a property called dissipativeness. Roughly speaking, this means that the system absorbs more energy from the external world than it supplies. Different types of dissipativeness can be considered depending on how one chooses to define "power input." Dissipativeness is shown to be characterized by the existence of a computable function which can be interpreted as the "stored energy" of the system. Under certain conditions, this energy function is a Lyapunov function which establishes stability, and in some cases asymptotic stability, of the isolated system.**

I. Introduction

In the study of a physical system, such as an electrical network or a mechanical machine, the concept of stored energy is often useful in deducing the behavior of the system. In many control problems, however, one is dealing with an abstract mathematical model where it may be difficult or even impossible to find some property of the model which corresponds to physical energy. In this short paper we show, for a certain class of nonlinear systems, that an "energy" approach can still be useful in stability analysis, despite the fact that the "energy" might not have any physical meaning. In effect, our technique is a method for generating Lyapunov functions.

The theory described here has its origins in work by Moylan and Anderson [1]–[3] and Willems [4] on the properties of passive systems. For our purposes, a passive system can be defined as a system which always dissipates energy, provided the energy input to the system is such that the product of system input and output represents input power. (This makes sense physically if the system is an electrical network [5]; in other cases, there might be no simple physical interpretation.) Several useful properties of passive systems have been noted in [2] for the linear case and in [1], [3] for nonlinear systems.

The notion of passivity was extended by Willems [4] to allow a more general definition of input power. A "dissipative system" is defined in [4] to be one for which a supply rate (input power) and storage function (stored energy) can be found, with the property that (in a sense made more precise in [4]) energy is always dissipated. As one might expect, there are some useful connections between dissipativeness (in Willems' sense)[1] and Lyapunov stability.

The present short paper extends the passivity results of [1] using concepts very similar to those of Willems [4]. Our approach differs from that of Willems primarily because we treat dissipativeness as an input–output property; that is, we do not postulate the existence of an internal storage function. However, we deduce the existence of such a storage function (in general nonunique), so it could be argued that this difference is unimportant. A more noticeable difference between our results and those of [4] is that, by sacrificing some generality, we obtain results which are considerably more explicit. The central result of this short paper is an algebraic criterion, in terms of functions of the system state, for the input–output property of dissipativeness. This result is then used to derive stability criteria.

The structure of the short paper is as follows. In Section II we derive the algebraic criterion for dissipativeness, which leads to computable storage functions. These functions have the properties of Lyapunov functions and in Section III are used to get stability results. In Section IV, the special case of passive systems is considered.

II. Algebraic Criterion for Dissipativeness

The systems to be studied are described by the equations

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) + J(x)u. \end{aligned} \tag{1}$$

The values of x, u, and y lie in R^n, R^m, and R^p, respectively. The admissable controls are taken to be locally square integrable. The functions $f: R^n \to R^n$, $G: R^n \to R^{n \times m}$, $h: R^n \to R^p$ of the state vector x satisfy $f(0) = 0$, $h(0) = 0$, and the following assumption.

Assumption 1: The functions appearing in (1) have sufficient smoothness to make the system well defined; that is, for any $x(t_0) \in R^n$ and admissible $u(\cdot)$, there exists a unique solution on $[t_0, \infty)$ such that $y(\cdot)$ is locally square integrable.

The above formulation includes a broad class of systems. In fact, Balakrishnan [7] has claimed that, under quite unrestrictive conditions, any time-invariant finite dimensional system can be represented by (1) with an appropriate choice of the state vector. However, it is not known whether the transformation in [7] will preserve our Assumption 1 and finite-dimensionality of the state space. Rather than explore these difficulties, we will adopt the attitude that the representation (1) is sufficiently general to warrant study in its own right.

Motivated by Willems [4], we associate with (1) a *supply rate*

$$w(u,y) = y'Qy + 2y'Su + u'Ru \tag{2}$$

where $Q \in R^{p \times p}$, $S \in R^{p \times m}$, and $R \in R^{m \times m}$ are constant matrices with Q and R symmetric. The supply rate is an abstraction of the concept of input power. In physical systems, input power is associated with the concept of stored energy. For more general abstract systems, such as given by (1), physical reasoning fails us, but we can define a possible

Manuscript received August 19, 1975; revised May 18, 1976. Paper recommended by G. L. Blankenship, Chairman of the IEEE S-CS Stability, Nonlinear, and Distributed Systems Committee. This work was supported by the Australian Research Grants Committee.

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[1]The word "dissipative" is used in an entirely different sense in some of the literature on differential equations (see, for example, [6]). Despite the coincidence in terminology, there is no connection between the two types of dissipativeness defined in [4] and [6]. Our definition is essentially equivalent to that of Willems [4].

candidate for the name "stored energy." Consider the function

$$\phi_a(x_0) = -\inf_{u(\cdot), T>0} \int_0^T w(u,y)\,dt \tag{3}$$

subject to (1) and $x(0)=x_0$. This is called the available storage in [4]; it can be interpreted as the maximum amount of energy which may be extracted from the system (1). Note that $\phi_a(x) \geq 0$ for all x.

In the sequel, we shall impose the following assumptions on the system (1) and the supply rate (2).

Assumption 2: The state space of the system (1) is reachable from the origin. More precisely, given any x_1 and t_1, there exists a $t_0 \leq t_1$ and an admissible control $u(\cdot)$ such that the state can be driven from $x(t_0)=0$ to $x(t_1)=x_1$.

Assumption 3: The available storage $\phi_a(x)$, when it exists, is a differentiable function of x.

Assumption 4: For any $y \neq 0$ there exists some u such that the supply rate (2) satisfies $w(u,y)<0$.

Assumptions 1–3 are not overly restrictive; in fact, for linear systems, Assumptions 1 and 3 are trivially satisfied. Assumption 4 is simply a restriction on the class of matrices Q, S, and R that will be considered. It is important to notice that u and y are considered to be independent variables—that is, they are not necessarily related by the state equations (1)—for the purposes of Assumption 4.

We now consider the concept of dissipativeness, which can be interpreted as saying that the initially unexcited system can only absorb energy. In this connection, it is worth noting that Assumption 4 ensures that the definition does not collapse to triviality.

Definition 1: The system (1) with supply rate (2) is said to be dissipative if for all admissible $u(\cdot)$ and all $t_1 \geq t_2$, we have

$$\int_{t_0}^{t_1} w(t)\,dt \geq 0 \tag{4}$$

with $x(t_0)=0$ and $w(t)=w[u(t), y(t)]$ evaluated along the trajectory of (1).

Two important special cases of dissipative systems are the following.

passive systems $\quad w=u'y$
finite gain systems $\quad w=k^2u'u-y'y$, k being a fixed scalar.

A trivial, but useful, observation is that if (1) is dissipative with respect to l supply rates w_i, $i=1,\cdots,l$, then it is also dissipative with respect to

$$w=\sum_{i=1}^{l} \alpha_i w_i$$

for any set $\{\alpha_i\}$ of nonnegative coefficients.

Notice that dissipativeness, as just defined, is an input–output property of the system. The following theorem, which is the central result of this short paper, shows that dissipativeness can also be characterized in terms of the coefficients in the state equation (1). A restricted version of this theorem, for passive systems only, appears in [1].

Theorem 1: A necessary and sufficient condition for (1) to be dissipative with respect to supply rate (2) is that there exist real functions $\phi: R^n \to R$, $l: R^n \to R^q$, and $W: R^n \to R^{q\times m}$ (for some integer q) satisfying

$$\begin{aligned} &\phi(x) \geq 0, \quad \phi(0)=0 \\ &\nabla'\phi(x)f(x) = h'(x)Qh(x) - l'(x)l(x) \\ &\tfrac{1}{2}G'(x)\nabla\phi(x) = \hat{S}'(x)h(x) - W'(x)l(x) \\ &\hat{R}(x) = W'(x)W(x) \end{aligned} \tag{5}$$

for all x, where

$$\hat{R}(x) = R + J'(x)S + S'J(x) + J'(x)QJ(x)$$

and

$$\hat{S}(x) = QJ(x) + S.$$

Proof: To prove sufficiency we suppose that $\phi(\cdot)$, $l(\cdot)$, and $W(\cdot)$ are given such that (5) is satisfied. Then for any admissible $u(\cdot)$, any t_0 and $t_1 \geq t_0$, and any $x(t_0)$, straightforward use of (5) gives

$$\int_{t_0}^{t_1} (y'Qy + 2y'Su + u'Ru)\,dt = \phi[x(t_1)] - \phi[x(t_0)] + \int_{t_0}^{t_1} [l(x)+W(x)u]'[l(x)+W(x)u]\,dt. \tag{6}$$

Setting $x(t_0)=0$, we have condition (4).

For necessity, we proceed to show that $\phi_a(\cdot)$ given by (3) is a solution of (5) for some appropriate functions $l(\cdot)$ and $W(\cdot)$.

For any state x_0 at $t=0$, there exists by Assumption 2 a time $t_{-1}<0$ and an admissible control $u(\cdot)$ defined on $[t_{-1},0]$ such that $x(t_{-1})=0$ and $x(0)=x_0$. From (4), then,

$$\int_0^T w(u,y)\,dt \geq -\int_{t_{-1}}^{0} w(u,y)\,dt.$$

The right-hand side of this inequality depends only on x_0, whereas $u(\cdot)$ can be chosen arbitrarily on $[0,T]$. Hence, there exists a function $C: R^n \to R$ of x such that

$$\int_0^T w(u,y)\,dt \geq C(x_0) > -\infty \tag{7}$$

whenever $x(0)=x_0$. From (7), we have $\phi_a(x)<\infty$ for all x. Also, dissipativeness implies that $\phi_a(0)=0$.

Now it is shown in [4] that ϕ_a satisfies

$$\phi_a(x_0) + \int_{t_0}^{t_1} w(u,y)\,dt \geq \phi_a(x_1)$$

for all $t_1 \geq t_0$ and all admissible $u(\cdot)$, where $x(t_0)=x_0$ and $x(t_1)=x_1$. Assumption 3 then gives

$$\frac{d\phi_a(x)}{dt} \leq w(u,y) \tag{8}$$

along any trajectory of (1). To turn this inequality into an equality, we introduce a function $d: R^n \times R^m \to R$ via

$$\begin{aligned} d(x,u) &= -\frac{d\phi_a(x)}{dt} + w(u,y) \\ &= -\nabla'\phi_a(x)[f(x)+G(x)u] + w[u, h(x)+J(x)u]. \end{aligned} \tag{9}$$

From (8), $d(x,u) \geq 0$ for all x and u. In addition, it is clear from (9) that $d(x,u)$ is quadratic in u. Combining these two observations, it follows that $d(x,u)$ may be factored as

$$d(x,u) = [l(x)+W(x)u]'[l(x)+W(x)u] \tag{10}$$

for some functions $l: R^n \to R^q$, $W: R^n \to R^{q\times m}$, and some integer q. (Notice, however, that the choice of q, l, and W is far from being unique.)

Substituting (10) into (9) gives

$$-\nabla'\phi_a(x)f(x) - \nabla'\phi_a(x)G(x)u + h'(x)Qh(x) + 2h'(x)\hat{S}(x)u + u'\hat{R}(x)u = l'(x)l(x) + 2l'(x)W(x)u + u'W'(x)W(x)u$$

for all x and u. Equating coefficients of like powers of u, we obtain (5) with $\phi=\phi_a$. △

Equation (6) can be interpreted as expressing an energy balance for system (1), and shows that the functions $\phi(\cdot)$ are storage functions satisfying Willems' definition [4]. The proof of Theorem 1 uses a different approach to that adopted in [1] for the special case of passivity. There the necessity part of the proof relied upon Hamilton–Jacobi theory. For a more complete discussion of dissipativeness along the lines of this short paper, the report [8] can be consulted. In particular, it is shown in [8] that the algebraic equations (5) possess maximum and minimum solutions which correspond to the *required supply* and *available storage*, defined in [4]. (Equation (3) provides the minimum solution. This observation will be useful in a later section.)

We can now give characterizations of dissipativeness, with respect to particular supply rates, by substituting the appropriate Q, R, S into (5).

Example 1: For finite gain, we set $Q=-I$, $S=0$, and $R=k^2I$ where k is a scalar. This gives (5) as

$$\nabla'\phi(x)f(x)=-h'(x)h(x)-l'(x)l(x)$$
$$\tfrac{1}{2}G'(x)\nabla\phi(x)=-J'(x)h(x)-W'(x)l(x)$$
$$k^2I-J'(x)J(x)=W'(x)W(x)$$

which, when specialized to linear systems, is a generalization of the Bounded Real Lemma [5].

A differential version of (6) is given by the following result.

Corollary: If system (1) is dissipative with respect to supply rate (2), then there exists a real function $\phi(\cdot)$ satisfying $\phi(x)\geqslant 0$, $\phi(0)=0$, such that

$$\frac{d\phi(x)}{dt}=-[l(x)+W(x)u]'[l(x)+W(x)u]+w(u,y) \qquad (11)$$

for the system (1).

Proof: This is simply a restatement of (9), with $d(x,u)$ defined as in (10). △

III. Stability of Dissipative Systems

Willems [4] suggests the usefulness of the theory of dissipative systems in the investigation of system stability via Lyapunov methods. The Lyapunov functions are generalized energy functions, corresponding to $\phi(\cdot)$ in Theorem 1. This leads us to consider conditions for which $\phi(\cdot)$ is positive definite, in the sense that $\phi(x)>0$ for all $x\neq 0$.

Definition 2: The system (1) is zero-state detectable if, for any trajectory such that $u(t)\equiv 0$, $y(t)\equiv 0$ implies $x(t)\equiv 0$.

Lemma 1: If the system (1) is dissipative with respect to supply rate (2) and zero-state detectable, then all solutions $\phi(\cdot)$ of (5) are positive definite.

Proof: The minimum solution of (5), given by (3), is positive definite if there exists a control such that $w(t)\leqslant 0$ on $[t_0,\infty)$, with strict inequality on a subset of positive measure. Now from Assumption 4, there certainly exists u such that $w(u,y)<0$ for any $y\neq 0$. In fact, the quadratic nature of $w(\cdot,\cdot)$ gives us sufficient freedom to choose such a u under the additional constraint that the chosen u and y be compatible with the state equations (1). (That is, if there exists any u such that $w(u,y)<0$, then—because $w(\cdot,\cdot)$ is quadratic in its arguments—there actually exists a wide choice of such u's. We could for example choose $u=K(x)y$, where there is enough freedom in choosing $K(x)$ to ensure that $[I-J(x)K(x)]$ is nonsingular for all x.) Accordingly, we can choose a feedback law $u^*(\cdot)$ such that $w(u^*(y),y)<0$ for $y\neq 0$, and $u^*(0)=0$.

We now have the desired result, provided that we can exclude the situation in which $y(t)=0$ for almost all t. However this last possibility is excluded by zero-state detectability. △

Now we easily obtain our main stability theorem.

Theorem 2: Let system (1) be dissipative with respect to supply rate (2) and zero-state detectable. Then the free system $\dot{x}=f(x)$ is (Lyapunov) stable if $Q\leqslant 0$ and asymptotically stable if $Q<0$.

Proof: From the corollary to Theorem 1 and Lemma 1, there exists a positive definite $\phi(\cdot)$ which satisfies

$$\frac{d\phi(x)}{dt}=-l'(x)l(x)+h'(x)Qh(x)$$

along the trajectories of $\dot{x}=f(x)$. The result then follows from standard Lyapunov stability [9] (for the case of $Q<0$, asymptotic stability follows by using a contradiction argument, based on the LaSalle Invariance Principle [9] and zero-state detectability). △

We see immediately that passive systems ($Q=R=0$, $S=I$) are stable and finite gain systems ($Q=-I$, $S=0$, $R=k^2I$) are asymptotically stable. That a finite gain system with a minimal state space is asymptotically stable has been shown in [10] using a different approach.

Theorem 2 has given conditions for the local stability of the equilibrium at $x=0$. It is evident that to achieve global asymptotic stability, we need to impose stronger conditions on the system. One way to do this is to assume that (1) is uniformly zero-state detectable, in the sense that there exists a strictly monotone increasing continuous function $\beta(\cdot)$ defined on $[0,\infty)$, with $\beta(0)=0$, and

$$\lim_{\sigma\to\infty}\beta(\sigma)=\infty$$

and a constant $T\geqslant 0$ such that for any x_0 and t_0 with $u(t)\equiv 0$

$$\int_{t_0}^{t_0+T}|h'Qh|\,dt\geqslant\beta(\|x_0\|).$$

This is similar to the definition of uniform observability used in [10], and it is easy to show that it implies that

$$\phi(x)\geqslant\beta(\|x\|)$$

for all solutions $\phi(\cdot)$ of (5).

IV. Stability of Passive Systems

As an illustration of the use of our theory in developing stability results for particular supply rates, we now confine our attention to passive stystems. As observed above and in [1], passive systems are stable. To achieve asymptotic stability, we introduce strong passivity—a property which ensures that no nontrivial trajectory is "free of dissipation of energy."

Definition 3: The system (1) is said to be 1) U-strongly passive (USP) if it is dissipative with respect to

$$w(u,y)=u'y-\epsilon u'u \qquad \text{for some } \epsilon>0,$$

2) Y-strongly passive (YSP) if it is dissipative with respect to

$$w(u,y)=u'y-\epsilon y'y \qquad \text{for some } \epsilon>0,$$

and 3) very-strongly passive (VSP) if it is dissipative with respect to

$$w(u,y)=u'y-\epsilon_1u'u-\epsilon_2y'y \qquad \text{for some } \epsilon_1>0,\ \epsilon_2>0.$$

One way to interpret these definitions is as follows. A YSP system is a passive system for which a small amount of positive feedback does not destroy the passivity property; a USP system can be similarly interpreted in terms of feedforward. (The usual definition [10], [11] of strict passivity corresponds to our definition of USP.) A VSP system is, of course, one which is both USP and YSP. Another example of the possibility of combining supply rates in the manner mentioned in Section II is provided by observing that a USP and finite gain system is VSP.

We now summarize stability results for passive systems which follow immediately from Theorem 2.

Theorem 3: For systems of the form (1), passive and USP systems are stable, while YSP and VSP systems are asymptotically stable.

The following example serves to illustrate the ideas of this short paper.

Example 2: We consider the equation

$$\ddot{x}+f(x)\dot{x}+g(x)=u \qquad (12)$$

where $f(\cdot)$, $g(\cdot)$ are functions of the scalar variable x and u is a forcing term. Setting $u=0$ gives the Lienard equation [9].

Letting $F(x)=\int_0^x f(\sigma)\,d\sigma$, a set of first-order equations equivalent to (12) is

$$\dot{x}_1=-F(x_1)+x_2$$
$$\dot{x}_2=-g(x_1)+u \qquad (13)$$

where $x_1=x$.

We combine (13) with the system output equation defined by

$$y=\alpha x_2-\beta F(x_1), \qquad \alpha\geqslant\beta\geqslant 0$$

and consider the passivity of this system (of course we could find conditions for other forms of dissipativeness). It is convenient to define $G(x)=\int_0^x g(\sigma)\,d\sigma$.

The calculations are straightforward and involve substitution into (5) and determination of conditions for the existence of a solution. We only present here the results for two special cases.

For $\alpha=1/2$, $\beta=0$, passivity follows if

$$G(x)\geqslant 0$$

and

$$g(x)F(x)\geqslant 0.$$

The storage function is

$$\phi_1(x)=\tfrac{1}{2}[\dot{x}+F(x)]^2+G(x).$$

For $\alpha=\beta=\frac{1}{2}$, passivity follows if

$$G(x)\geqslant 0$$

and

$$f(x)\geqslant 0.$$

The storage function is

$$\phi_2(x)=\tfrac{1}{2}\dot{x}^2+G(x).$$

The functions ϕ_1 and ϕ_2 are standard Lyapunov functions used for the study of the stability of the Lienard equation; choice of ϕ_2 as a Lyapunov function is motivated by its interpretation as the sum of the kinetic and potential energies of (12), and $\phi_1=\phi_2+\dot{x}F(x)+\frac{1}{2}F(x)^2$ is called the modified energy function [9]. We have now shown that they can both be interpreted as stored energy functions (depending on how one defines the system output) arising from a study of the passivity of (13).

V. Conclusion

The main result presented here is Theorem 2, which relates the stability of a broad class of nonlinear systems to the input–output property of dissipativeness. We can characterize dissipativeness by the existence of a computable function $\phi(\cdot)$ of the state. This function is thought of as the stored energy of the system and under certain conditions is a Lyapunov function.

Our approach can also be applied profitably to interconnected systems, in particular feedback systems. One can, for example, derive Lyapunov versions of the stability criteria of Zames [11], using a Lyapunov function which is the sum of the storage functions for the individual subsystems. Details of these and other results are currently in preparation and will be reported separately.

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Identification of Linear Systems with Time-Delay Operating in a Closed Loop in the Presence of Noise

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Abstract—**The subject of this short paper is on-line identification of the parameter vector defining a linear dynamical system which operates in a closed loop in the presence of noise, and incorporates a time delay. The method is based on the equation error. Other known subsystems in the closed-loop system increase the dimension of the closed-loop parameter vector which tends to degrade the estimation convergence process. By means of "composite state variables," introduced in this short paper, this increase is prevented and the open-loop parameters are directly identified from closed-loop input-output data. The pure time delay in the closed loop causes a representation problem in the equation error formulation. This is overcome by "composite delayed state variables." The value of the time delay is determined by means of excess parameters provided by a "higher order model" and a simple on-line search procedure. The method is illustrated by simulated examples.**

I. Introduction and Statement of the Problem

The subsystem $G(s)$ of unknown structure and order to be identified includes a pure time delay and is part of a closed-loop system incorporating other known dynamical subsystems (Fig. 1). The system is excited by a given stationary random input, and Gaussian uncorrelated additive noise is assumed to be present in the loop. In terms of Laplace transforms, $G(s)$ is given by

$$G(s)=e^{-\tau s}\frac{N(s,\boldsymbol{b})}{D(s,\boldsymbol{a})}=\frac{e^{-\tau s}\sum_{i=0}^{m}b_is^i}{1+\sum_{j=1}^{n}a_js^j}\tag{1}$$

where m, n denote the highest numerator and denominator powers, respectively, and $m\leqslant n$. $G(s)$ operates in a closed loop (Fig. 1) with other dynamical elements $G_1(s)$, $G_2(s)$ and $G_3(s)$ having known parameters. Tracking or regulating tasks in manual control are special cases of the system in Fig. 1. The following assumptions regarding the input, system, and noise are made.

Assumption 1: The input $x(t)$ is a sample of a given random stationary mean square bounded ergodic process. Its spectral distribution guarantees a persistent excitation of all the modes of $G(s)$.

Assumption 2: $G(s)$ and the closed-loop system denoted by $T(s)$ is stable and time invariant.

Assumption 3: The noise $n_I(t)$ (Fig. 1) is a zero-mean stationary ergodic Gaussian process uncorrelated with $x(t)$.

It is required to provide unbiased estimates of $\boldsymbol{a}$ and $\boldsymbol{b}$ from $x(t)$ and a suitably chosen closed-loop system output. Since the sensitivity of the closed-loop system output [output of $G_2(s)$] to variations in $\boldsymbol{a}$, $\boldsymbol{b}$ is reduced by the loop gain, the system error, denoted by $z(t)$ (Fig. 1) which retains this sensitivity, is chosen as the appropriate system output. The corresponding closed-loop transfer function relating $x(t)$ to $z(t)$ is

$$T(s)=G_1(s)[1+G_1(s)G_2(s)G_3(s)G(s)]^{-1}.\tag{2}$$

In time domain, $z(t)$ is given by

$$z(t)=T(p)x(t)+n(t)=y(t)+n(t)\tag{3}$$

where $p^i\triangleq d^i/dt^i$, $i=0,1,2,\cdots$, and

$$n(t)=-G_2(p)G_3(p)G(p)T(p)n_I(t).\tag{4}$$

$y(t)$ is the system error in the absence of noise. The known subsystems

Manuscript received August 12, 1975; revised April 1, 1976. Paper recommended by E. Tse, Past Chairman of the IEEE S-CS Stochastic Control Committee.

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