2) $\int_{-\infty}^{0} q(x,u)\,dt = 0$ for all $x, u \in \mathcal{L}^2_{(-\infty,0]}$.

3) $\Phi(s) = 0$.

Moreover, $\Lambda(P) \geq 0$ has a unique solution P and by necessity $P \leq 0$ and $\Lambda(P) = 0$.

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On a Frequency-Domain Condition in Linear Optimal Control Theory

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Abstract—**A simple frequency domain inequality is known to be necessary but not, in general, sufficient, for the existence of the infimum defined by a linear-quadratic optimal control problem. This note shows that for a certain class of performance indices, the condition is also sufficient.**

In a recent paper, Willems [1] established a number of existence results for linear-quadratic optimal control problems. Subsequently, one of these results (relating a frequency-domain inequality to a time-domain inequality) was found to be incorrect [2]. However, it remains plausible that the result in question is correct for some important subclasses of the problem treated in [1]. The purpose of this note is to point out the existence of one such subclass.

The problem treated in [1] was the following: Given the linear system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{1}$$

and the performance index $\int_0^\infty w(x,u)\,dt$, under what conditions is the performance index bounded below? The loss functions were of the form

$$w(x,u) = x'Qx + 2u'C'x + u'Ru$$

with Q and R being symmetric matrices. However, there was no requirement that, for example, Q should be nonnegative, or that R should be nonsingular. Specifically, [1] considered the relationships between the following inequalities (among others).

1) The boundedness condition (BC):

$$V_f^+(x_0) = \inf_u \int_0^\infty w(x,u)\,dt > -\infty.$$

(Here the subscript f refers to a free endpoint problem.)

2) The linear matrix inequality (LMI):

$$\begin{bmatrix} A'K + KA + Q & KB + C' \\ B'K + C & R \end{bmatrix} \geq 0$$

for some $K = K' \leq 0$.

3) The frequency domain inequality (FDI):

$$R + C(sI - A)^{-1}B + B'(s^*I - A')^{-1}C' + B'(s^*I - A')^{-1}Q(sI - A)^{-1}B \geq 0$$

for all complex s in $\operatorname{Re} s \geq 0$.

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4) The dissipation condition (DC):

$$\int_0^T w(x,u)\,dt \geq 0$$

for every $T \geq 0$ and every pair x, u constrained by (1) and by $x(0) = 0$.

It was concluded in [1] that (given controllability) the DC, the LMI, and the BC were equivalent to one another. It was also shown that any of these three implied the FDI. Unfortunately, the converse is not true [2]; an interesting open question, then, is to determine classes of $w(\cdot\cdot)$ for which the FDI does in fact imply the remaining conditions.

One such class was provided by Willems in [2].

Lemma 1: Suppose that $w(x,u)$ can be written as $w(x,u) = y_1'y_1 - y_2'y_2$, where $y_1 = C_1x + D_1u$, $y_2 = C_2x + D_2u$, and suppose further that the matrix $W_1(s) = D_1 + C_1(sI - A)^{-1}B$ is square and invertible. Then, given controllability of (1), the FDI implies the LMI.

Proof: With $W_1(s)$ defined as above, and $W_2(s) = D_2 + C_2(sI - A)^{-1}B$, the FDI may be written as $W_1'(s^*)W_1(s) - W_2'(s^*)W_2(s) \geq 0$. With $W_1(s)$ invertible, this may be rewritten as $I - [W_2(s^*)W^{-1}(s^*)]'[W_2(s)W_1^{-1}(s)] \geq 0$, for all s in $\operatorname{Re} s \geq 0$. The result then follows from known results for scattering matrices [3]. □

It will now be shown that the result holds for another important class of $w(\cdot\cdot)$, namely those satisfying the following assumption.

Assumption 1: For any x, there exists some u such that $w(x,u) \leq 0$.

Some consequences of this assumption were pointed out in [1]. In particular, Assumption 1 implies that $V_f^+(x) \leq 0$ for all x.

Lemma 2: Suppose that (1) is completely controllable and that $w(x,u)$ satisfies Assumption 1. Then the FDI implies the LMI.

Proof: Assume first that R is nonsingular. Then we have

$$w(x,u) = x'(Q - C'R^{-1}C)x + (u + R^{-1}Cx)'R(u + R^{-1}Cx).$$

Clearly, the FDI implies that $R \geq 0$; Assumption 1 then implies that $Q - C'R^{-1}C \leq 0$. Now writing $Q - C'R^{-1}C = -C_2'C_2$ for some C_2, it is seen that $w(x,u)$ is in a form such that Lemma 1 may be applied, and the result is proved.

If R is singular, the FDI implies a similar inequality, with R replaced by $R + \epsilon I$ for *any* $\epsilon > 0$. Using the result for nonsingular R, we then have that the LMI is satisfied, with R replaced by $R + \epsilon I$. This in turn implies the dissipation condition

$$\int_0^T [w(x,u) + \epsilon u'u]\,dt \geq 0$$

for all $T \geq 0$, whenever $x(0) = 0$. Since this is true for any $\epsilon > 0$, a contradiction argument may be used to show that it is also true for $\epsilon = 0$. Using the known equivalence between the DC and the LMI, the result is proved. □

The results of this work may be summarized as follows.

Theorem: Let the system (1) be completely controllable, and suppose that $w(\cdot\cdot)$ satisfies Assumption 1. Then the following hold.

1) The BC, the LMI, the FDI, and the DC are pairwise equivalent.

2) If any of the conditions is satisfied, then $-\infty < V_f^+ \leq 0$.

The proof of these assertions follows easily from Lemma 2 and the results in [1].

A further interesting point is that if Assumption 1 is strengthened to require the existence of $u(x)$ such that $w(x,u(x)) \leq 0$ *and* such that $\dot{x} = Ax + Bu(x)$ is asymptotically stable, then the following hold.

1) *Every* symmetric solution of the LMI satisfies $K = K' \leq 0$.

2) The FDI may be replaced by a simpler test on $\operatorname{Re} s = 0$ (rather than $\operatorname{Re} s \geq 0$).

These observations follow easily from the known results in [1], and so will not be expanded on here.

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